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Life expectancy of physical property based on mortality laws

Edwin Bernard Kurtz
Iowa State College

LIFE EXPECTANCY OF PHYSICAL PROPERTY

BY

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LIFE EXPECTANCY OF PHYSICAL PROPERTY

BASED ON MORTALITY LAWS

By

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PREFACE

The engineer and accountant concerned with valuation, appraisal, and depreciation problems hitherto have had available for use no absolutely dependable tables of life expectancy for physical property. Such data as have been organized are admittedly opinions as to the probable average period of usefulness of different items of equipment. At best they are but the assumptions of the engineering and accounting professions.

Strange as it may seem, few mortality data of physical property have been collected and presented in actuarial form. The lack of such data for structures, machinery and equipment has been commented upon frequently. Three notable statements to this effect are:

J. E. Allison, Commissioner and Chief Engineer of the St. Louis Public Service Commission, in a paper dated September, 1912:

"An examination of the estimates of different species of property as used in most published valuation reports will show that there is comparatively close agreement in the lives assigned to similar items of equipment. This might indicate that these estimates were the result of reliable statistics of experience gathered throughout the whole field where the class of equipment in question is used. As a matter of fact no such broad collection of reliable data has ever been made, and these estimates of average life admittedly represent only the opinions of certain men as to the probable average period of usefulness of different items of equipment. The fact that they coincide within certain limits only goes to show that the later guessers did not differ very greatly from their predecessors on a subject concerning which there was very little to be found to support an argument one way or another."

C. E. Grunsky, Consulting Civil Engineer, San Francisco, California, in his book, "Valuation, Depreciation and the Rate Base," published in 1917:

"Unfortunately there are no records available from which absolutely dependable tables of expectancy could be prepared for each class of perishable articles in use in connection with public service properties, such as have been prepared by actuaries for human beings."

Report of the Committee of the National Association of Railway Commissioners on "Plans for Ascertaining Fair Valuation of Railroad Property," dated 1912:

"One of the most suggestive items of information gathered in the analysis of these data (comparative description of appraisal methods) is the variation in blanket or arbitrary percentages adopted for a few of the different items of the appraisal. It has been somewhat of a disappointment that our interrogatories have failed, with few exceptions, to bring out the facts supporting these percentages. They are evidently assumptions of the engineering profession; but it would seem wise, especially in view of the wide disparity in the percentages, for the Committee in the future to follow up this matter, and, if possible, make a somewhat critical analysis of the actual facts justifying the percentages which have been adopted."

Recognizing this absence of mortality data, some fourteen years ago the author began to search out and compile such records as could be secured. As a result, a group of mortality tables has been assembled, to include almost all data which have come to his attention during the search. Undoubtedly, the information for many more such tables exists in the files of industrial and appraisal engineers, public and industrial accountants, and in the records of public service commissions, utilities, and industries. As the value and usefulness of a knowledge of the life characteristics of physical property become more manifest, it is quite likely that many additional data will be made available.

The study and analysis to which these various mortality tables have been subjected, have produced this book. The life characteristics of actual physical property were calculated and analyzed and such relations as could be readily discovered were noted. This is the most fundamental and valuable part of this treatment as it is the first time that these relations have been set forth for physical property. Many of these relations may prove to be laws, and so permit of a scientific determination of the life expectancy of equipment and its attending problems. In any event, the data presented are available for use and cover a wide range of structures and machinery.

It is confidently believed that these basic findings will in time become the foundation stones on which theories and practices of depreciation and valuation will be erected as superstructures. Hardly a single contribution has appeared in all history on which the diverse and voluminous discussions in these fields could rest for support. Further discussion henceforth will have to recognize the fundamental discoveries herein presented; in fact, every depreciation accountant and appraisal or valuation engineer will have to become familiar with the life characteristics of the classes of property with which he deals, in order intelligently and properly to carry on his work. Moreover, the facts and relations presented herein will furnish a common background to all views, for they must constitute the beginning of all sound thinking in this field.

Nearly all of the data from which the mortality tables and curves were constructed were obtained from outside sources. Acknowledgment is here made for the use of the material, to the following: Administration; American Electric Railway Association; American Society of Civil Engineers; American Water Works Association; American Wood Preservers' Association; Archiv Für Post und Telegraphie; H. L. Baker, City Engineer, Chicago; Electric Company of Missouri; Illuminating Engineers Society; The Milwaukee Electric Railway and Light Company; National Electric Lamp Association; New York Telephone Company; Railway Age; United States Forest Products Laboratory.

The author acknowledges his indebtedness to Edwin Gruhl and F. W. Doolittle, and is especially grateful for inspiration and assistance to Dean Anson Marston under whose guidance and direction this study was made.

EDWIN B. KURTZ

Iowa City, Iowa,
May 24, 1930.

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BASED ON MORTALITY LAWS

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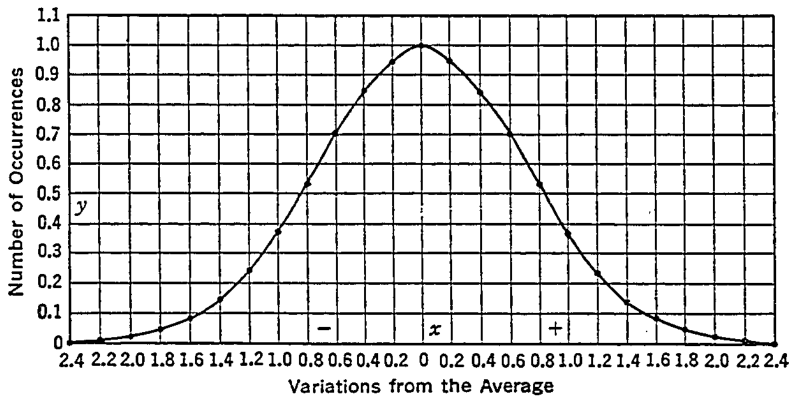
Life Expectancy of Property.—Depreciation and obsolescence are two wasting processes that are under continual study in business and in industry. Hitherto no one method of determining these factors has been universally accepted. In fact, many methods have had advocates and partisans. Further, there has been no agreement as to the reasonable life expectancy for various classes of physical property such as structures, machinery, and equipment, although agencies concerned with valuation, appraisals, and corporation income taxes have made extensive investigations and prepared numerous compilations.

In recent months several surveys have been undertaken to determine the condition of the physical equipment of several basic American industries. The thought behind these efforts is that much of the producing equipment in industries which have not been prosperous, is obsolete and out of date. Difficulties which have seemed insurmountable are the setting up of a criterion for obsolescence; and finding a measure for the productive life of an item of equipment.

From an appreciation of this situation, and a realization of the pressing need of a new attack upon the broad problems of determining life expectancy in advance, this volume has been undertaken. The main objectives have been three: to establish mortality tables of physical property on an actuarial basis; to develop the life characteristics of different classes of physical property; to develop the relations, and determine the laws, between the various life characteristics.

It is evident that the first purpose must be accomplished before the remaining objectives can be carried through to a conclusion. However, strange as it may seem, very few mortality data on physical property have been collected and put into actuarial form.

There now follows a presentation of the development of mortality tables of physical property, prepared in a similar



$Y = k\frac{1}{e^{h^2x^2}}$, where $k = 1$, $h = 1$ and $e = 2.73$

Figure 1. Theoretical Normal Distribution Curve

form to those which have been accepted as reliable for human mortality.

Theoretical Probability Curve.—The application of the "theory of probabilities" to the probability of occurrence of different magnitudes of variations from the average in an infinite number of observations of equal scientific care, or of occurrences equally likely to be exactly the same, gives rise to the theoretical probability (or distribution) curve shown in Figure 1. The equation of this curve is as follows:

$$Y = k\frac{1}{e^{h^2x^2}} \qquad \text{where } k = 1$$

$$h = 1$$

$$\text{and } e = 2.73$$

The theoretical curve of Figure 1 indicates the greater probability of the occurrence of small than of large variations from the average of a large number of observations or other occurrences. Manifestly, it cannot be applied in more than a suggestive way in the study of variations from the average life of

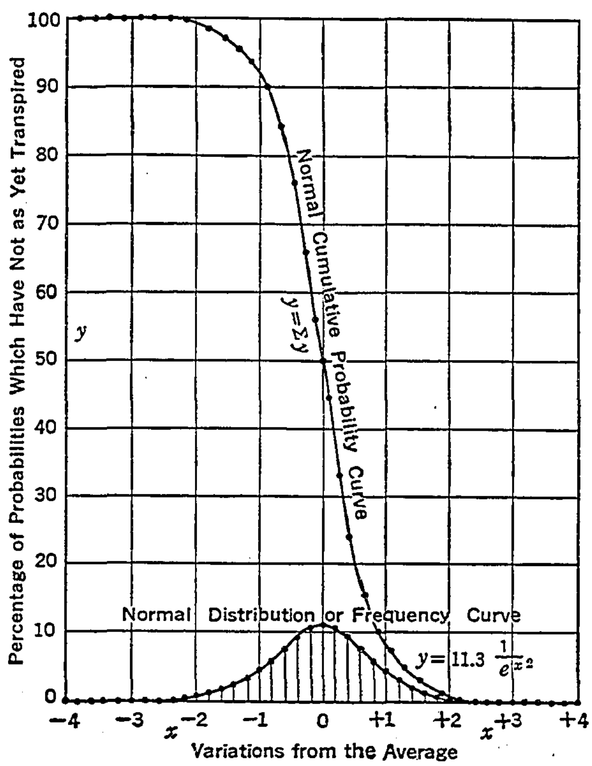


Figure 2. Normal Probability Curves

the lives of the individual units of physical property. However, the actual distribution and mortality curves of physical properties, plotted from life experience records, show some strikingly general resemblances. This resemblance can be noted by comparing the distribution and cumulative curves of Figure 2 with the survivor and frequency curves of actual physical property shown in Chapters 3 and 4.

Table 1. United States Mortality Table for Males in Original Registration States—1910

Age Interval in Years	Of 100,000 Males Born Alive		Expectation of Life in Years	Total Probable Life in Years
	Number Dying in Age Interval	Number Alive at Beginning of Age Interval		
(1)	(2)	(3)	(4)	(5)
0– 1	12,495	100,000	49.86	49.86
1– 2	2,521	87,505	55.94	56.94
2– 3	1,108	84,984	56.59	58.59
3– 4	676	83,876	56.33	59.33
4– 5	482	83,200	55.79	59.79
5– 6	395	82,718	55.11	60.11
6– 7	333	82,323	54.37	60.37
7– 8	383	81,990	53.59	60.59
8– 9	243	81,707	52.77	60.77
9– 10	215	81,464	51.93	60.93
10– 11	196	81,249	51.07	61.07
11– 12	189	81,053	50.19	61.19
12– 13	190	80,864	49.30	61.30
13– 14	199	80,674	48.42	61.42
14– 15	214	80,475	47.54	61.54
15– 16	233	80,261	46.66	61.66
16– 17	260	80,028	45.80	61.80
17– 18	291	79,768	44.95	61.95
18– 19	325	79,477	44.11	62.11
19– 20	360	79,152	43.29	62.29
20– 21	396	78,792	42.48	62.48
21– 22	422	78,396	41.70	62.70
22– 23	431	77,974	40.92	62.92
23– 24	433	77,543	40.14	63.14
24– 25	435	77,110	39.37	63.37
95– 96	93	289	2.36	97.36
96– 97	66	196	2.25	98 25
97– 98	46	130	2.13	99.13
98– 99	31	84	2.02	100.02
99–100	20	53	1.91	100.91
100–101	14	33	1.81	101.81
101–102	8	19	1.70	102.70
102–103	5	11	1.60	103.60
103–104	3	6	1.51	104.51
104–105	1	3	1.41	105.41
105–106	1	2	1.32	106.32
106–107	1	1	1.23	107.23

The theoretical probability curve first published in connection with mortality curves of physical property, so far as the writer is aware, was by Edwin Gruhl.[1]

The Human Mortality Curve.—Reliable mortality tables and curves must be based upon extensive statistics of the actual lives of many individual units rather than upon any general

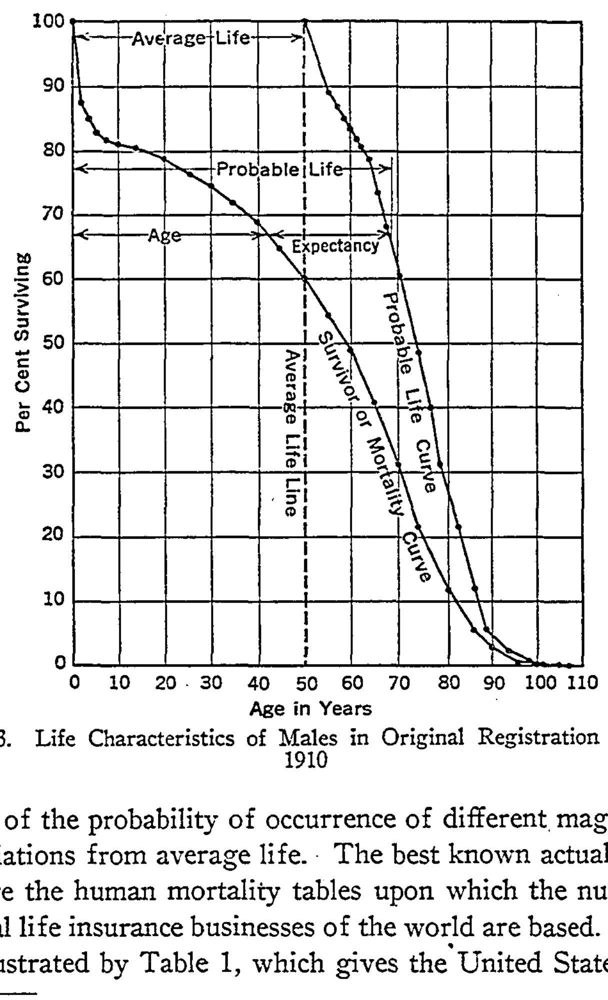


Figure 3. Life Characteristics of Males in Original Registration States—1910

theory of the probability of occurrence of different magnitudes of variations from average life. The best known actual examples are the human mortality tables upon which the numerous colossal life insurance businesses of the world are based. These are illustrated by Table 1, which gives the United States mor-

[1] A. E. R. A., March, 1913, published by the American Electric Railway Association.

tality data for males only. From such a table one can compute the average life of a person at age 0, the future expectancy at any age, and the total probable life at any age. This has been done, and the results are tabulated in columns (4) and (5).

Table 1 contains the data for the plotting of the life characteristics of United States males, shown in Figure 3. In this figure the horizontal distance from the ordinate scale to the mortality curve represents age, the horizontal distance from the mortality curve to the probable life curve represents the expectancy at that age, and the total horizontal distance from the left edge to the probable life curve, which is the sum of age plus expectancy, represents the probable life.

It is of interest to note that the future expectancy of humans is greatest at 2 years and actually is greater at age 10 than at age 0. This is due to the large infant mortality, and is not true of units of physical property. As a general rule, the future expectancy gradually decreases with age, as is apparent in Figure 3 by the gradual decrease in the horizontal distance between the mortality curve and the probable life curve with advance in age. The probable life is an increasing function with age, which is shown by the gradual slope of the curve to the right.

The human mortality table and human life characteristics will be different for different countries, for different periods in the same country, and for persons of the same race engaged in different occupations. Marked differences also exist in the tables constructed for males and for females. Figure 4, taken from the United States Life Tables, shows the mortality curves for various races and countries and brings out these differences very strikingly.

Development of Mortality Tables of Physical Property.—The use of mortality tables and curves in determining the life characteristics of physical property seems so natural as almost to be inevitable. Lack of the necessary statistical life

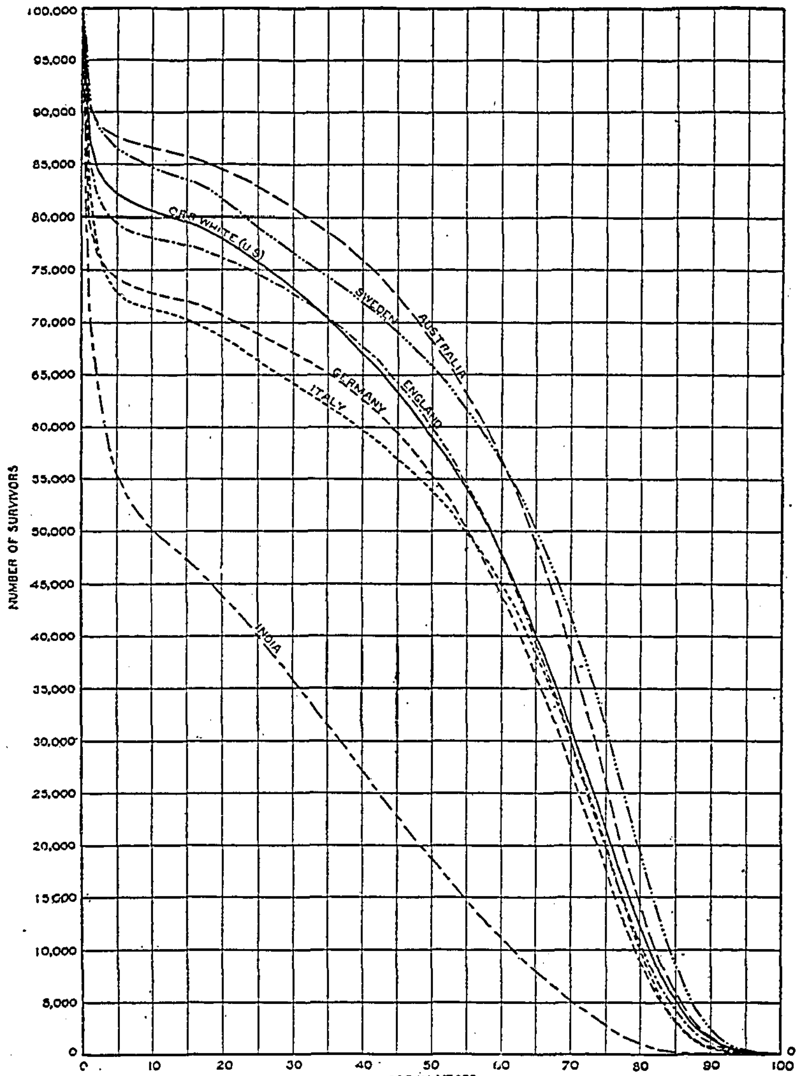


Figure 4. Survivor Curves for Humans for the United States and Various Foreign Countries. (Reprint from United States Life Tables.)

experience data, however, has prevented the compilation of extensive mortality tables of physical property. The various

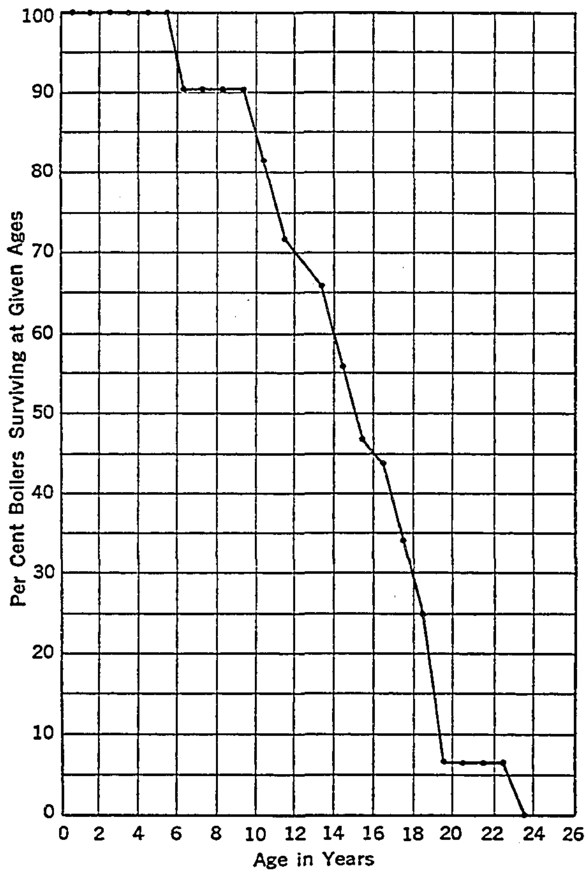


Figure 5. Survivor Curve of Waterworks Boilers. (Plotted from John W. Alvord's Data of 1903.)

steps and contributions leading to our present knowledge are briefly reviewed in the following paragraphs:

1903—**John W. Alvord.**—In 1903, John W. Alvord published tables [2] of the life experience of 48 waterworks pumps

[2] Proceedings of the American Waterworks Association, pp. 479, 484.

and 32 waterworks boilers. The majority of the pumps were retired because they were inadequate, a few were discarded because they had become obsolete, and the remainder were scrapped. Of the 32 boilers several were still in use, and a

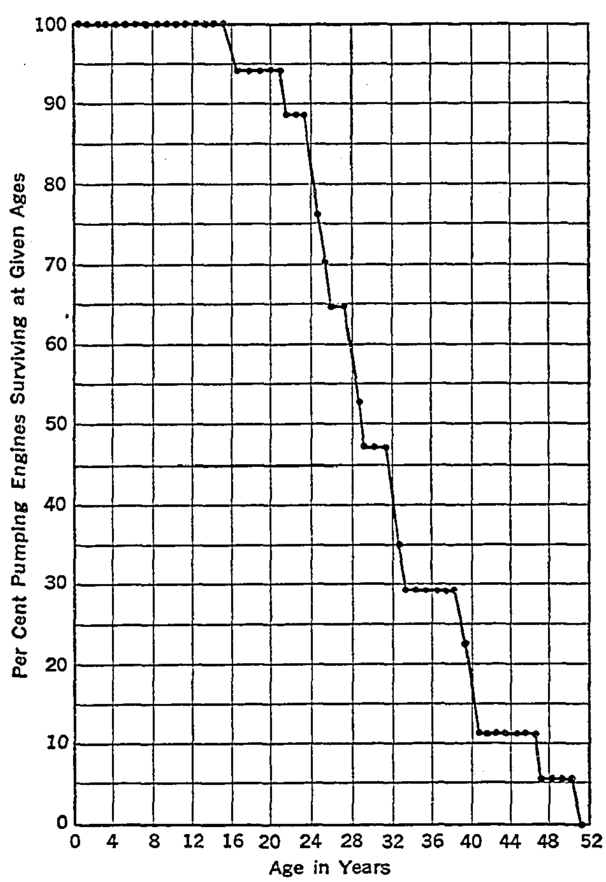


Figure 6. Survivor Curve of Waterworks Pumping Engines. (Plotted from John W. Alvord's Data of 1903.)

number were still in place but not in active use. The data in the tables plat in the curves shown in Figures 5 and 6, which would have been the first mortality curves of physical property published (so far as the author is aware) if the data had been presented in graphical as well as in tabular form. To the author's knowledge Mr. Alvord did not make any attempt to develop a mortality table or to utilize the data he had secured to calculate expectancies and probable lives after the analogy

Table 2. Relative Life and Value of Wooden Poles

(By Gen. Ober-postrat Christiani, Berlin. Taken from *Archiv Für Post und Telegraphic*, Nr. 16, August, 1905.)

Age in Years	Treatment					Total
	$CuSO_4$ (1)	$ZnCl_2$ (2)	Coal Tar (3)	$HgCl_2$ (4)	Untreated (5)	
1..........	392	637	106	7	6	1,148
2..........	2,063	1,802	220	85	53	4,223
3..........	5,248	4,699	435	187	123	10,692
4..........	6,327	5,988	823	435	393	13,966
5..........	6,366	7,854	1,024	707	682	16,633
6..........	5,591	9,042	1,705	1,141	732	18,211
7..........	4,913	9,851	1,833	1,487	927	19,011
8..........	5,147	9,911	2,184	1,094	924	19,260
9..........	4,759	11,337	2,889	1,425	499	20,909
10..........	4,629	10,871	3,114	1,094	171	19,879
11..........	4,676	11,586	3,100	978	424	20,764
12..........	3,895	7,848	2,861	736	114	15,454
13..........	3,561	7,433	2,721	492	30	14,237
14..........	4,522	6,788	2,177	292		13,779
15..........	3,207	4,438	1,933	165	21	9,764
16..........	3,122	4,007	1,131	130	144	8,534
17..........	2,821	3,752	905	151	30	7,659
18..........	2,620	3,349	550	174	225	6,918
19..........	2,085	1,955	151	190	210	4,591
20..........	862	796	35	105		1,798
21..........	524	241	41	9		815
22..........	146	105	62			313
23..........	83	10	9			102
24..........	47					47
25..........						
Totals.......	77,606	124,300	30,009	11,084	5,708	248,707

of human mortality tables. It should be pointed out that the table on boilers was prepared on the basis that all the boilers had lived their complete useful life. The actual expectancies would therefore be somewhat greater.

1905—Gen. Ober-postrat Christiani.—In August, 1905, there appeared an article by Mr. Christiani, entitled "Relative Life and Value of Wooden Poles." [3] As a basis for comparison, Mr. Christiani presented a table, shown here as Table 2, giving the life experience of 248,707 wooden poles. The poles were classed in groups according to the preservative treatment used as shown in the table. These statistics go back to 1852 and include the experience of the North-German and even the Prussian telegraph system, thus covering a period of more than 50 years. From these data Mr. Christiani calculated the average life of the different kinds of poles and drew conclusions as to the economic value of the different treatments.

It is of interest to note the opinion Mr. Christiani held relative to the value of the statistics shown. He said:

"During 52 years, 4,659,816 telegraph poles of different kinds have accordingly been under observation. Such a long period of observation and such an extraordinarily large number of observations which have occurred under the most varying local conditions have permitted the calculation of mean lives which can lay claim to general validity.

"The restriction to a single line or to a shorter period of observation would afford no guarantee for the reliability of the average figures. For on the one hand the life of poles depends for the same kind of treatment to a large degree on the dimensions as well as on the species, on the age, and on the conditions of growth of the tree from which the poles are obtained; on the other hand, on the character of the soil in which they set, and on the climatic influences to which they are exposed. The diversity of conditions could not but make itself felt in a small series of observations; it, however, is compensated if we can give the inquiry as broad a scope as was at our command for the calculations in question."

The foregoing data are therefore among the earliest life statistics of physical property. Mr. Christiani, however,

[3] *Archiv Für Post und Telegraphie*, Nr. 16.

omitted to calculate expectancies and probable life at all ages. His only calculation was at zero age.

The mortality curves resulting from these statistics are shown in Figure 7.

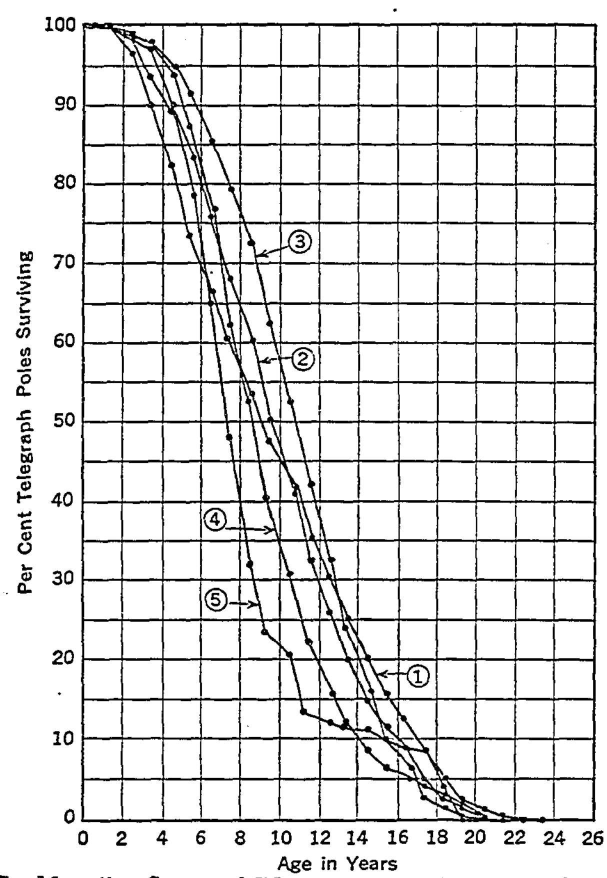


Figure 7. Mortality Curves of Wooden Telegraph Poles. (See mortality data given in Table 2.)

1911—American Railway and Maintenance of Way Association.—In 1911 a subcommittee of the American Railway and Maintenance of Way Association on Statistics of Cross Ties issued a report [4] based on the returns from a questionnaire

[4] Volume 12, Part I, 1911, p. 359.

which was sent to all railroad companies and which read as follows: "Enclosed herewith is a blank form which can easily be filled out by giving the year the ties were laid, the number of ties removed each year to date, and the kind of ties used, whether treated or untreated."

One of the replies received was from George E. Rex, manager of treating plants of the Santa Fé Railway System. This reply brought out the general situation and is quoted in part below:

"For years we have attempted to keep our record of ties on every section of track on the road; but with the class of help we get on this work it seems almost impossible to get an accurate record, and for this reason we discontinued this method and picked out experimental sections on each Division Superintendent's division, taking an actual inventory of the ties in the track and all other information that would be of value, and then make an annual inspection of each one of these sections and have monthly reports made of ties taken out and put in. This we hope will in time give us an absolute record of the life of our ties."

The committee compiled the data received and made it a part of its report. The data are those of some of the largest roads in the country and go as far back as 1897 in some cases. The period covered is thus from 1897 to 1910.

The general conclusion reached by the committee was, that few railroads, if any, have complete or reliable information concerning the life of ties, but that most railroad companies have good records of renewals of cross ties each year, and in many cases keep such records by miles.

1912—National Electric Lamp Association.—On December 15, 1912, there was published by the National Electric Lamp Association the mortality curve of incandescent lamps shown in Figure 8.[5] The curve summarizes the life experience

[5] Bulletin of Engineering Research on Lamp Efficiency, p. 23.

of incandescent lamps in use at that time as obtained from periodic inspections. The curve therefore illustrates mortality due only to use and does not represent any replacements due to inadequacy, obsolescence, or public requirement. The mortality curve shown has a very definite shape, being almost identical to one-half of a cosine curve. The manufacture of lamps

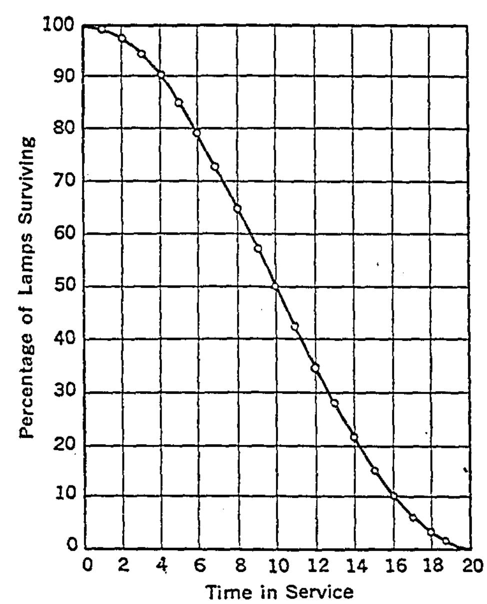


Figure 8. Mortality Curve of Incandescent Lamps in Use in 1912. (Note similarity to cosine curve.)

has greatly improved since 1912 and no cosine curve will fit the mortality data of lamps in use today. For comparison, see Figure 9, which shows the life experience of Mazda B lamps as found by the National Electric Lamp Works.[6]

1913—Edwin Gruhl.—In 1913, Edwin Gruhl, at that time assistant to the president of the Milwaukee Electric Railway & Light Company but now general manager of the North

[6] Bulletin No. 13-F, pp. 7-9 inclusive, dated December 5, 1917.

American Company, published a paper on "Depreciation Estimates" in the journal of the American Electric Railway Association. In this paper Mr. Gruhl published a mortality curve for waterworks pumps from John W. Alvord's data of 1903 (Figure 6), the N. E. L. A. cosine curve for incandescent lamps (Figure 8), and in addition, a mortality curve showing

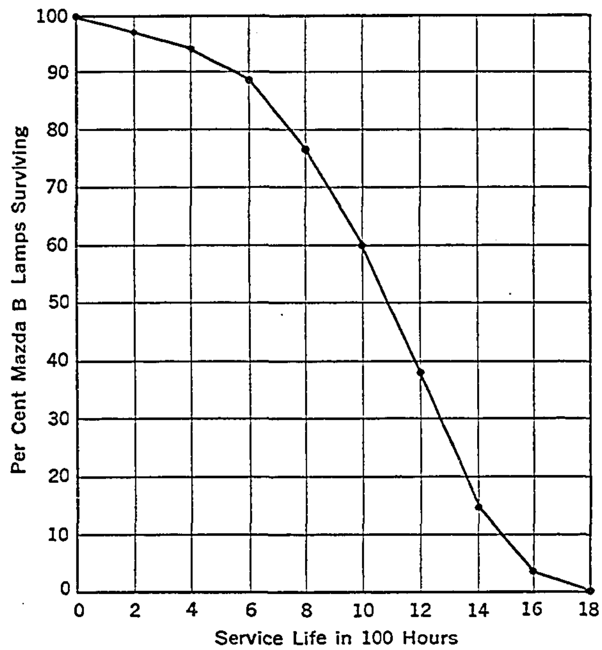


Figure 9. Survivor Curve of Mazda B Lamps of 1917

the life expectancy of cast iron wheels shown in Figure 10. The life experience of car wheels summarizes the experience for the year 1910 on the Milwaukee Street Railway System and represents 939 out of a total of 3,304 wheels removed. Of the total, 47% of the wheels were removed because worn out, 27% because of chipped flanges, and 26% because of flat wheels. The curve therefore represents the mortality due to replacements from accidental causes as well as wearing out in service.

Mr. Gruhl also called attention to the similarity of the above named mortality curves to the theoretical normal probability curve and to the human mortality curve. This portion of Mr. Gruhl's work was extremely valuable as an early step in the development of the use of mortality curves of physical prop-

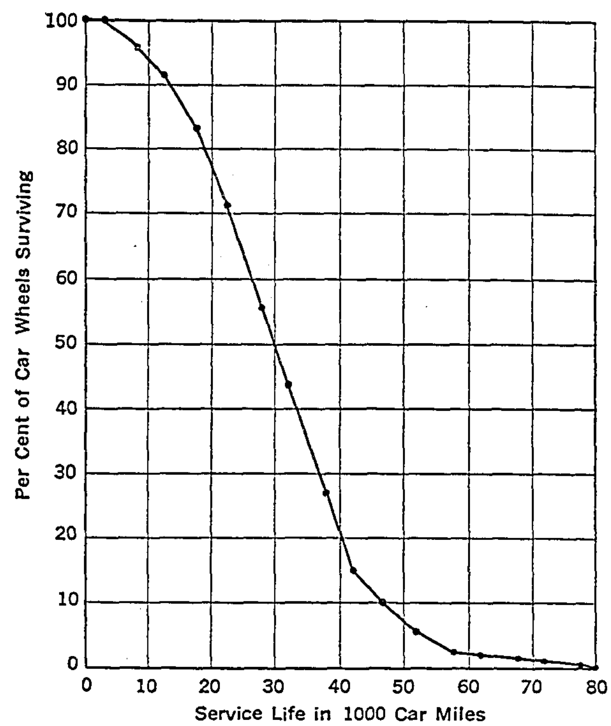


Figure 10. Life Experience of 939 Cast Iron Car Wheels. (As Found by Edwin Gruhl in 1913.)

erty. Mr. Gruhl, however, did not compute average lives, life expectancies, or probable lives, which quantities he could have readily used in estimating depreciation or per cent condition. On the contrary, Mr. Gruhl suggested using the ordinates to the mortality curve to represent per cent condition. On this basis he compared the per cent condition as derived by the straight line, sinking fund, and mortality curve methods.

1916.—At the request of the author, H. L. Baker, assistant city engineer of Chicago, prepared a table showing the life experience of pumping engines removed during the last 50 years

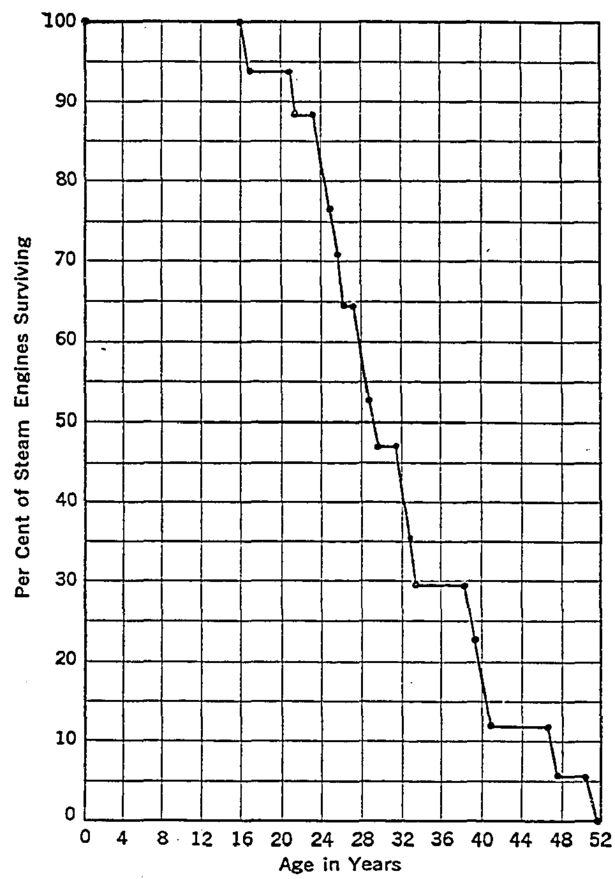


Figure 11. Life Experience of Pumping Engines in City of Chicago. (Curve is based on 17 units.)

in the city of Chicago. This tabulation is shown in Table 3 and the mortality curve obtained therefrom is shown in Figure 11. It is interesting to note that although there are only 17 units in the group the mortality curve is quite smooth and typical.

"The engines at the Chicago Avenue and 22nd Street stations were of the vertical walking beam type, while practically all the others were horizontal Gaskill or Worthington engines. In practically all cases they could have been run for an indefinite period, but the cost of repairs and operation was high, and the requirements of greater capacity and greater economy caused their replacement."

Table 3. Tabulation of Life of Pumping Engines—Chicago Waterworks
(Data collected by John A. Elger from Annual Repairs)

Location of Pumping Station	Date of Installation	Date Removed	Life in Years	Cause of Removal
Chicago Ave...	1853	1904	51	Obsolete—Replaced by modern engine
" ...	1857	1904	47	"
" ...	1867	1906	39	"
" ...	1872	1905	33	"
22nd Street....	1876	1916 (To be removed)	40	"
"	1876	"	40	"
"	1884	"	32	"
"	1884	"	32	"
68th Street.....	1888	1916	28	"
"	1888	1916	28	"
"	1890	1916	26	"
"	1892	1916	24	"
"	1892	1916	24	"
Lake View.....	1884	1913	29	"
"	1888	1913	25	"
"	1892	1913	21	"
"	1897	1913	16	"

These life experience data of pumping engines were not published until July, 1921, when they appeared in an article by the author.[7]

1916—New York Telephone Co.—In the year 1916, Sergius P. Grace, representing the New York Telephone Co.,

[7] Edwin B. Kurtz, "Replacement Insurance," in *Administration.*

introduced four exhibits showing the life experience of aerial, underground, and submarine cable, in a rate case before the New Jersey Board of Public Utility Commissioners. These four exhibits are shown in Figures 12, 13, 14, and 15.

The method used in the compilation of the data shown in these four figures is illustrated in Table 4 and Figure 16. The

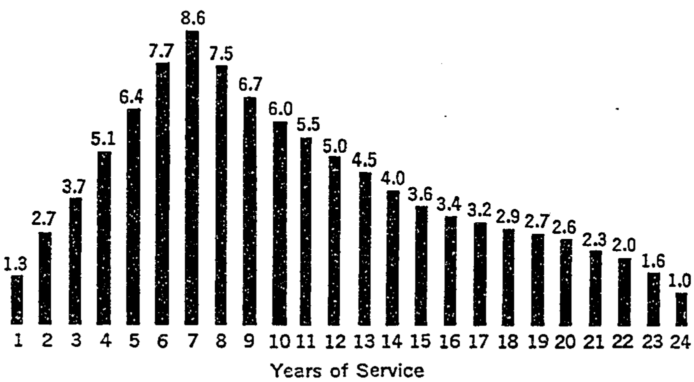


Figure 12. Life of Aerial Cable

(Exchange and Toll Excluding Composite and Coarse Gauge Cable—Westchester Division)

Vertical Lines Show Per Cent of Cable Displaced Each Year

1. Value in service 1-1-17	$ 968,429	
2. Removals accounted for to 1-1-17	323,890	
3. Total installation accounted for to 1-1-17	1,292,319	
4. Ratio of removals to total installed	25.0%	
5. Ratio of removals to remainder in service	33.4%	
6. Age of oldest cable now in service	20	years
7. Average age of cable now in service	6.2	"
8. Average age of cable removed to 1-1-17	5.4	"

Consideration of the cable actually retired in relation to that capable of retirement, indicates:

9. A probable maximum age of	24	years
10. A probable average age of	10.6	"

method involved obtaining the ratio between (a), the amount of plant actually retired up to and including each of N years of service, and (b), the total amount of plant which was capable of retirement after the same number of years of service. The column headings in Table 4 show the steps employed in obtaining this relation. The graph shows the accumulative per cent of plant displaced plotted against years of service.

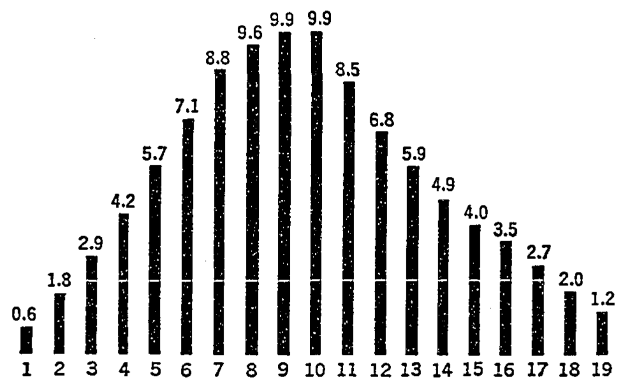


Figure 13. Life of Aerial Cable

(Exchange and Toll Excluding Composite and Coarse Gauge Cable—New Jersey Division Including Staten Island)

1. Value in service 1-1-16.................................. $2,108,299
2. Removals accounted for to 1-1-16........................ 386,910
3. Total installation accounted for to 1-1-16.................. 2,495,209
4. Ratio of removals to total installed........................ 15.5%
5. Ratio of removals to remainder in service.................. 18.4%
6. Age of oldest cable now in service.......................... 16 years
7. Average age of cable now in service........................ 3.44 "
8. Average age of cable removed to 1-1-16.................. 6.35 "

Actually retired in relation to that capable of retirement:

9. A probable maximum age of.............................. 19 years
10. A probable average age of................................ 9.7 "

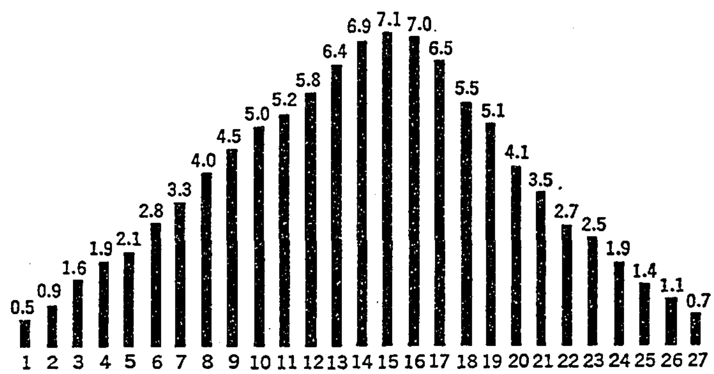


Figure 14. Life of Underground Cable

(Main—Exclusive of Coarse Gauge—Manhattan and Bronx)

1. Value in service 1-1-15.................................. $5,826,873
2. Removals accounted for to 1-1-15........................ 1,433,484
3. Total installation accounted for to 1-1-15.................. 7,260,357
4. Ratio of removals to total installed (2/3).................. 19.7%
5. Ratio of removals to remainder in service (2/1)............ 24.6%
6. Age of oldest cable now in service.......................... 26 years
7. Average age of cable now in service........................ 7.2 "
8. Average age of cable removed to 1-1-15.................. 8.5 "

Actually retired in relation to that capable of retirement:

9. A probable maximum age of.............................. 27 years
10. A probable average age of................................ 14.2 "

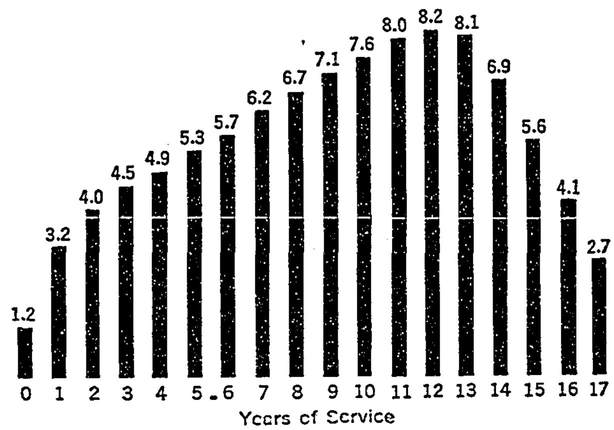


Figure 15. Life of Submarine Cable

(RUBBER AND LEAD SHEATH—MANHATTAN AND BRONX)

VERTICAL LINES SHOW PER CENT OF CABLE DISPLACED EACH YEAR

1. Value in service 1-1-15	$135,851	
2. Retirements accounted for to 1-1-15	330,332	
3. Total installation accounted for to 1-1-15	466,183	
4. Ratio of retirements to total installed (2/3)	70.9%	
5. Ratio of retirements to remainder in service (2/1)	243.2%	
6. Age of oldest cable in service 1-1-15	15	years
7. Average age of cable in service 1-1-15	5.7	"
8. Average age of cable retired to 1-1-15	8	"

Consideration of the cable actually retired in relation to that capable of retirement, indicates:

9. A probable maximum age of	17	years
10. A probable average age of	9.3	"

These exhibits, with the exception of the studies made by Mr. Christiani in 1905, are undoubtedly the most comprehensive mortality studies of physical property made up to that time. It should be noted, however, that no expectancies or probable lives or other analyses were made. Only the average life of each group was computed.

Table 4. Table Showing Length of Service of Aerial

Year Placed	Years Since Invest.	Value* of Plant Placed Still in Service	Value* of Plant Placed Since Removed	Total Value* of Plant Placed	Accum. Total Value* of Plant Placed	Value of							
						N= 0	1	2	3	4	5	6	7
1895	20		354	354	354								
1896	19		2,443	2,443	2,797								
1897	18		9,945	9,945	12,742								
1898	17		14,404	14,404	27,146			19					
1899	16	3,339	24,260	27,599	54,745			43	496	148		124	2,853
1900	15	2,022	30,292	32,314	87,059			184		75	383	1,910	5,257
1901	14	2,791	11,797	14,588	101,647			301	15	80	1,226	2,684	1,370
1902	13	5,660	33,248	38,908	140,555				120	1,639	3,403	5,165	6,918
1903	12	10,611	35,243	45,854	186,409			19	2,889	4,932	4,474	4,094	4,470
1904	11	5,475	21,327	26,802	213,211			1,671	1,263	1,480	5,021	3,511	2,148
1905	10	12,681	19,021	31,702	244,913		785	1,968	914	2,660	3,044	1,134	1,820
1906	9	47,623	47,722	95,345	340,258	242	2,260	4,489	7,435	9,376	4,430	3,566	10,181
1907	8	97,586	29,346	126,932	467,190	552	1,225	1,552	3,116	2,981	4,415	6,949	7,524
1908	7	69,718	23,620	93,338	560,528		689	1,286	3,010	5,679	4,555	7,742	659
1909	6	138,151	28,792	166,943	727,421		351	4,745	6,129	4,659	11,800	1,108	
1910	5	229,397	24,923	254,320	981,791	135	1,590	2,067	8,360	8,475	4,296		
1911	4	187,637	10,495	198,042	1,179,833		794	1,645	6,687	1,297			
1912	3	348,705	7,626	356,331	1,536,164	149	421	3,658	3,398				
1913	2	412,262	10,533	422,795	1,958,959	43	5,611	4,879					
1914	1	444,877	1,609	446,486	2,405,445	470	1,139						
1915	0	89,764		89,764	2,495,209								
Total		2,108,299	386,910	2,495,209									
(1) Total value displaced						1,591	14,865	28,526	43,832	43,463	47,047	37,987	43,200
(2) Gross accumulated value displaced						1,591	16,456	44,982	88,814	132,277	179,324	217,311	260,511
(3) Deduct displacements of plant less than N Years of age (accumulated value)								1,609	12,142	19,768	30,173	55,096	83,888
(4) Net accumulated value displaced (2)–(3)						1,591	16,456	43,373	76,672	112,509	149,151	162,215	176,623
(5) Accumulated total value of plant placed						2,495,209	2,405,445	1,958,959	1,536,164	1,179,833	981,791	727,421	560,528
(6) Accumulated % displaced (4)–(5)						0.06	0.68	2.2	5.0	9.5	15.2	22.3	31.5
(7) % displaced each year (difference from curve)†						0	0.6	1.8	2.9	4.2	5.7	7.1	8.8

* Reproduced value new obtained by applying 1-1-15 unit costs (as determined for appraisal purposes) to units of plant placed

† The percentage figures on line (7) are read from a smooth curve below plotted against the points on line (6).

Cable—New Jersey Division (Including Staten Island)

Plant Removed After N Years of Service

8	9	10	11	12	13	14	15	16	17	18	19	20	21	Total
										354				354
	22	227	340			623	433	685		113				2,443
	737	5,959	339	80	1,417	40	117	959	297					9,945
1,624	3,775	2,105	2,250	2,501	1,100	243	515	127	145					14,401
1,370	1,998	1,703	7,515	1,657	1,780	2,970	902	701						24,260
4,289	2,388	5,287	3,977	1,078	3,034	2,261	169							30,292
989	955	663	1,246	873	1,395									11,797
2,940	2,592	3,330	2,563	4,020	558									33,248
4,708	5,567	2,263	1,682	145										35,243
1,827	2,410	1,713	283											21,327
2,355	3,674	667												19,021
4,048	1,695													47,722
1,032														29,346
														23,620
														28,792
														24,923
														10,405
														7,626
														10,533
														1,609
25,182	25,813	23,917	20,195	10,354	9,284	6,137	2,136	2,472	442	467				386,910
285,693	311,506	335,423	355,618	365,972	375,256	381,393	383,529	386,001	386,443	386,910	386,910	386,910	386,910	
107,508	136,854	184,576	203,597	224,924	260,167	293,415	305,212	335,504	359,764	374,168	384,113	386,556	386,910	
178,185	174,652	150,847	152,021	141,048	115,989	87,978	78,317	50,497	26,679	12,742	2,797	354	0	
467,190	340,258	244,913	213,211	186,409	140,555	101,647	87,059	54,745	27,146	12,742	2,797	354		
38.1	51.3	61.6	71.4	75.7	81.8	85.5	90.0	92.3	98.2	100.0	100.0	100.0		
9.6	9.9	9.9	8.5	6.8	5.9	4.9	4.0	3.5	2.7	2.0	1.2			

and retired.

May 1, 1917.
Data as of 1-1-16.

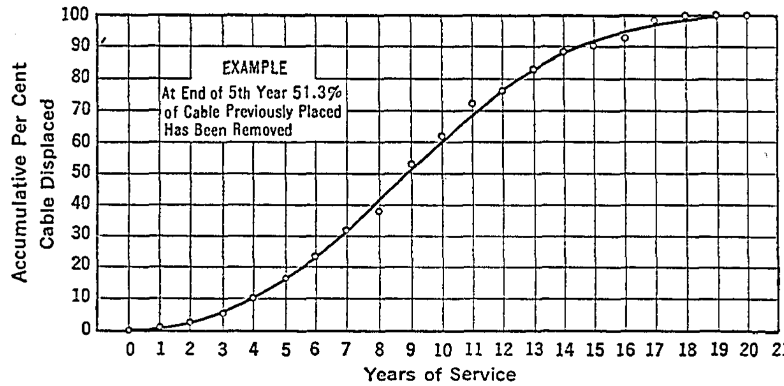


Figure 16. Accumulative Per Cent Cable Displaced Plotted Against Years of Service. (Points on this curve are plotted from column 6 of Table 4.)

1917.—In 1917, the author, then a senior engineering student at the University of Wisconsin, submitted a thesis entitled, "Depreciation and Replacement Insurance of Physical Property." In this paper were included 17 mortality tables, the largest collection of life experience of physical property brought together up to that date. Eleven of these mortality tables have already been discussed in the foregoing paragraphs. The remaining six tables not yet referred to include five mortality tables of railroad rolling stock to be discussed hereafter and one mortality table giving the life history of 1,372 electric poles (Figure 17). The data on electric poles were obtained from the Electric Company of Missouri and cover the period from 1910-1915 inclusive. Of these 1,372 poles, 42% were removed because of butt rot, 31.2% because too short, 0.8% because not heavy enough, 0.6% because of accidents, and 25.4% because of street or civic improvements. The table thus includes such causes of replacement as decay, casualties, and public requirement. These data have not been published hitherto.

In this thesis the author advocated actual replacement insurance as a scientific plan for making provision for replacements of physical property in an industrial enterprise. In the plan the amount of insurance of any given unit of property is made equal to the cost of the unit. The annual premium which the company should set aside each year into an interest-bearing fund is then dependent upon the cost, the interest rate, and the

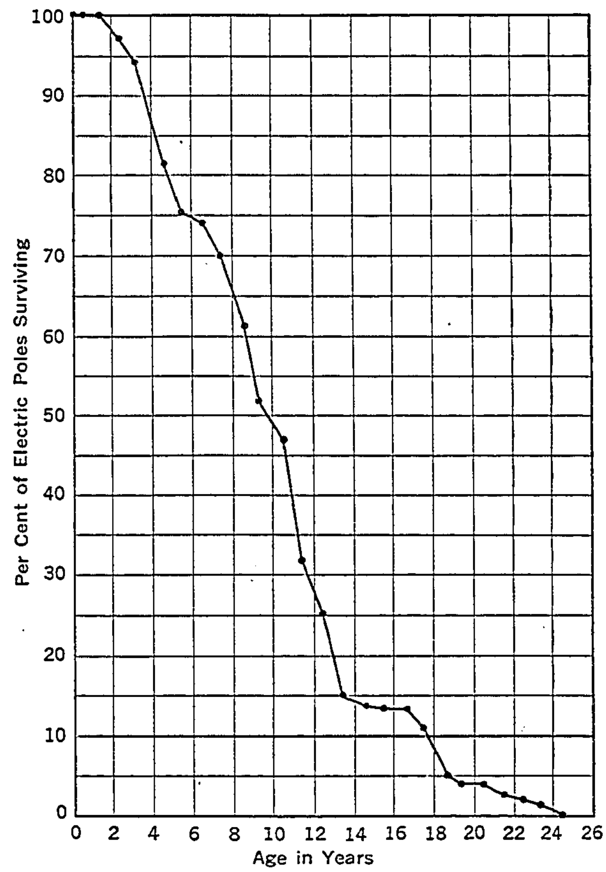


Figure 17. Survivor Curve of 1,372 Electric Poles of the Electric Company of Missouri

probability of the unit going out of service. The latter is obtained from mortality experience. The sum of the premiums of all the units of property in service constitutes the total amount to be put in a reserve each year to provide for replacement. As a unit is retired, the benefit is used for the purchasing of a new unit. The property thus becomes perpetual. The plan is proposed as a substitute for the present arbitrary methods of providing for depreciation. In the article referred to, the method of applying the actuarial method is outlined, and actual calculations for pole data are shown.

In this thesis the author also calculated average lives and expectancies used in the calculation of per cent condition, thus carrying the use of mortality data considerably beyond any previous attempts.[8]

Table 5. Assumption as to Uniform Increase and Decrease of Failures of Original Units

(From "Valuation, Depreciation and the Rate Base," by C. E. Grunsky)

Year or Period	For 10,000 Articles			Single Article
	Number of Failures	Remaining Number of Articles at Beginning of Year	Remaining Service Years at Beginning of Year	Expectancy at Beginning of Year or Period
1	100	10,000	100,000	10.00
2	200	9,900	90,000	9.09
3	300	9,700	80,100	8.27
4	400	9,400	70,400	7.46
5	500	9,000	61,000	6.77
6	600	8,500	52,000	6.12
7	700	7,900	43,500	5.51
8	800	7,200	35,600	4.95
9	900	6,400	28,400	4.44
10	1,000	5,500	22,000	4.00
11	900	4,500	16,500	3.67
12	800	3,600	12,000	3.33
13	700	2,800	8,400	3.00
14	600	2,100	5,600	2.67
15	500	1,500	3,500	2.33
16	400	1,000	2,000	2.00
17	300	600	1,000	1.67
18	200	300	400	1.33
19	100	100	100	1.00
20	0	0	0	0

[8] The thesis was published in part in July, 1921, in *Administration*, pp. 41-70, under the title "Replacement Insurance."

1917—**C. E. Grunsky.**—In 1917, C. E. Grunsky, a consulting engineer, published a book entitled "Valuation, Depreciation and the Rate Base." This book has already been quoted in the preface in connection with the lack of mortality statistics of physical property. In the absence of actual life experience data, Mr. Grunsky assumed various possible distributions of replacements, chief among which are those shown in Tables 5 and 6. The hypothesis assumed in Table 5 is that the replacements increase uniformly from year to year up to the average life, after which they decrease uniformly. The table shows the increase of renewals from 100 to 1,000 at average life, and the

Table 6. Assumption of Failures of Original Units in Accordance with Law of Probability

(From "Valuation, Depreciation and the Rate Base," by C. E. Grunsky)

Year	For 10,000 Articles			Single Article
	Number of Failures	Remaining Number of Articles, Beginning of Year	Remaining Service Years, Beginning of Year	Expectancy at Beginning of Year
1	15	10,000	100,000	10.00
2	35	9,985	90,000	9.00
3	85	9,950	80,015	8.05
4	180	9,865	70,065	7.11
5	330	9,685	60,200	6.22
6	550	9,355	50,515	5.40
7	805	8,805	41,160	4.78
8	1,065	8,000	32,355	4.04
9	1,265	6,935	24,355	3.52
10	1,340	5,670	17,420	3.08
11	1,265	4,330	11,750	2.71
12	1,065	3,065	7,420	2.42
13	805	2,000	4,355	2.18
14	550	1,195	2,355	1.97
15	330	645	1,160	1.80
16	180	315	515	1.64
17	85	135	200	1.50
18	35	50	65	1.25
19	15	15	15	1.00
20	0	0	0	0

subsequent decrease from 1,000 to zero. In Table 6 the annual replacements are assumed to be in accordance with the law of probability. The table shows the annual replacements based on this assumption increasing from 15 to 1,340 and the corresponding decrease to zero. In each table are also shown the expectancies for each age of property life. All of the data of Table 5 are shown graphically in Figure 18. By comparing this assumed mortality curve with those shown in Chapter 3,

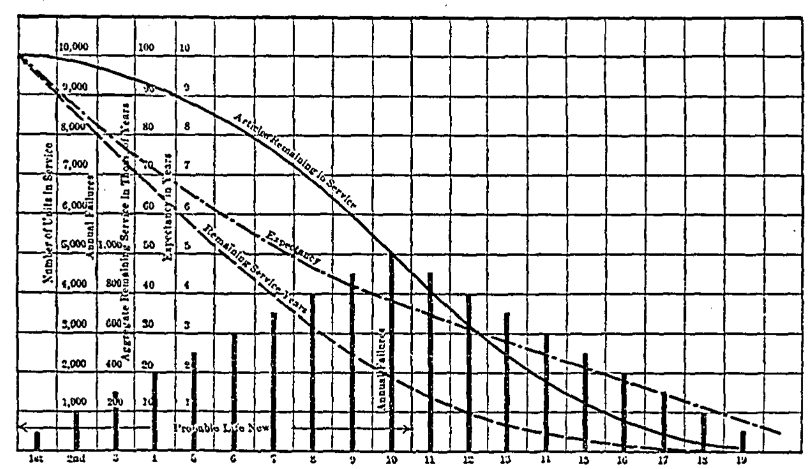


Figure 18. Diagram Showing Data of Table 5 Plotted Against Years

it will be seen that Mr. Grunsky's assumption was quite stimulating.

Mr. Grunsky made additional calculations based on these assumed distributions, such as remaining value, and replacement requirements. These are fully discussed in Chapter 6 of his book, "Actual and Probable Life."

In justification of these assumptions and calculations Mr. Grunsky comments on page 107 as follows:

"These are presented not only as an improvement in the method of estimating probable remaining life or expectancy of any article which is no longer new, though still in good condition, but also to encourage further study along these lines, in order that where necessary, closer approximation of the actual accrued and annual depreciation can be made than has heretofore been attempted."

1917—Valuation Committee, American Society of Civil Engineers.—In 1917 a special committee of the American Society of Civil Engineers submitted its final report on "Principles and Methods for the Valuation of Railroad Property and Other Public Utilities." [9] Appendix II of this report is entitled, "Some Examples of the Expectation of Life of So-called Permanent Structures." Three examples of structures are given, namely, railway stations, water supply sources, and pumping stations. The life experience of a number of units was given for each type of property. Table 7 shows the form

Table 7. Life of Pumping Stations

Location of Pumping Station	Period, Years	Life in Years
Boston, original high service	1870–1888	18
Old East Boston	1880–1889	9
New East Boston	1889–1898	9
West Roxbury	1896–1913	27
Mystic, Worthington Pump	1864–1898	34
Mystic, Leavitt Pump*	1896–1898	2
Somerville	1890–1900	10
Malden, Spot Pond Station	1883–1898	15
Malden, Webster Park Station	1890–1900	10
Chelsea	1886–1900	14
Everett	1888–1899	11
Quincy	1884–1898	14
Hyde Park, Neponset River Station	1885–1912	27
Hyde Park, Mother Brook Station	1899–1912	13
Medford, Spot Pond Station	1892–1898	6
Medford, Reservoir Station	1894–1899	5
Revere, Town Station	1884–1898	14
Revere, Cliftondale Station	1891–1899	8
Melrose	1886–1899	13
Watertown	1885–1898	13
Arlington	1895–1900	5
Swampscott	1885–1899	14
Lexington	1884–1903	19

* This pumping engine was transferred to another pumping station when the Mystic Station was abandoned.

[9] Transactions of 1917, pp. 1311-1620

in which the data were presented for pumping stations. Similar tables were given for the other types of structures mentioned.

The only use made of these data by the committee was to determine the average life of the units in each group. The committee called attention to the relatively short life lived by many of the units tabulated, and concluded by saying:

"These examples have been cited merely to indicate that functional depreciation is an active force to be considered and given such practical weight as circumstances and experience may indicate to be fair."

1918—Mable E. Thorne.—In 1918 there was published a paper by Mable E. Thorne entitled, "Relation between Average Life of Ties and Percentage of Renewals." [10] Miss Thorne was then statistician for the Forest Products Laboratory of the United States Forest Service. This study of average life and renewals was based on the renewal records collected from a number of railroads. A typical record as obtained from the railroad is given in Table 8.

Forty-three such records comprising a total of 42,936 cross ties were available and were used in the study. Table 9 is a composite of these 43 records giving the cumulative per cent ties removed against per cent average life. The treated and untreated ties are shown in separate columns.

Table 8. Typical Report of Cross Tie Life Experience

(Number ties set, 521; locality, Central States; average life, 8.9 years; preservative, zinc chloride; species, hemlock.)

Years Service	Per Cent Removed	Per Cent of Average Life
6	0	67
7	20	29
8	36	90
9	55	101
10	98	112
15	100	169

[10] Proceedings of the American Wood-Preservers' Association.

Table 9. Ties Removed at Intervals of 5% of Average Life as Determined by a Combination of 43 Different Records of 42,936 Ties

Per Cent of Average Life	(a) Untreated	(b) Treated	Per Cent of Average Life	(a) Untreated	(b) Treated
1–25	0	0	120	91.4	90.9
30	0	0	125	92.8	93.4
35	0.4	0.1	130	93.9	95.9
40	1.2	0.5	135	94.7	96.6
45	2.0	0.6	140	94.9	97.4
50	2.9	1.6	145	95.3	97.9
55	3.8	2.2	150	95.8	98.2
60	5.7	2.8	155	96.1	98.3
65	8.3	5.2	160	96.6	98.6
			165	96.9	98.9
70	11.1	10.5			
75	15.0	17.6	170	97.4	99.1
80	23.6	24.3	175	98.0	99.5
85	36.6	33.1	180	98.6	99.9
90	41.3	43.0	185	99.0	99.9
95	55.2	52.9	190	99.3	99.9
100	63.6	60.8	195	99.6	99.9
105	76.6	67.4	200	99.7	99.9
110	85.7	74.6			
115	98.4	84.1			

(a) Based on 26 records comprising 12,185 ties.
(b) Based on 17 records comprising 30,751 ties.

The same data are shown in graphical form in Figure 19. Miss Thorne pointed out that the similarity in renewals of the various records led her to combine the records in an attempt to show the general law or relation of the renewals in any group of ties to the life of the group. According to Miss Thorne such a curve could be used:

1. To estimate the average life of a group of ties long before all the ties in the group had been removed.
2. To estimate the expected renewals in any subsequent year after the average life had been determined.
3. To check up on the practice of various railroads or divisions of the same road.
4. To serve as a basis for a table of depreciation values.

Miss Thorne's work is of importance for it lends further strength to the mortality method. The life experience of ties shown above includes enough units to make the data representative.

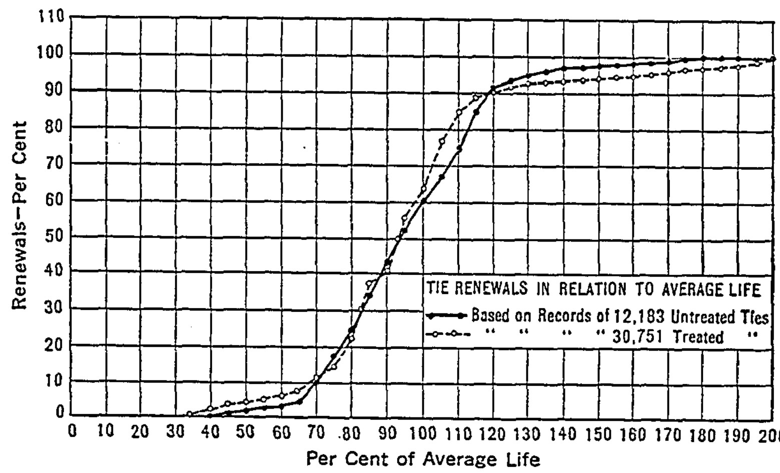


Figure 19. Cross Tie Renewals in Relation to Average Life

1918—E. J. Kates.—E. J. Kates in 1918 published an article entitled, "The Actuary Theory of Depreciation of Physical Property Values."[11] He presented three mortality tables of railroad rolling stock. Table 10 gives Mr. Kates' life experience data on locomotives. The data plat in the curves shown in Figure 20. Similar tables were given for passenger cars and freight cars. The table on freight cars was a composite curve giving the combined life experience on coal, flat, box, stock, and refrigerator cars. The life experience data were obtained by Mr. Kates from the records of practically all the railroads west of Chicago. They are therefore a composite from these various roads and cover an extended period of time.

Mr. Kates used these data for the calculation of life expectancies for each year of property life. From the values of life

[11] *Railway Age*, May 3, 1918.

Table 10. The Kates Depreciation Studies for Locomotives

Years in Service	Per Cent Remaining in Service	Expectancy in Years of Service	Expectancy in Per Cent of Remaining Service	Expectancy in Per Cent of Remaining Value
0	100	25.4	100	100
1	100	24.4	96.1	96.5
2	100	23.4	92.1	92.9
3	100	22.4	88.2	89.4
4	100	21.4	84.3	85.6
5	100	20.4	80.3	82.3
6	100	19.4	76.4	78.8
7	100	18.4	72.5	75.3
8	100	17.4	68.5	71.6
9	99.5	16.5	64.7	68.2
10	99	15.6	60.9	64.8
11	98	14.7	57.2	61.5
12	97	13.8	53.5	58.2
13	95.5	13.0	50.0	55.0
14	94	12.2	46.6	51.9
15	91	11.6	43.6	49.2
16	88	11.0	40.8	46.7
17	84	10.4	38.0	44.2
18	80	9.9	35.5	42.0
19	75	9.5	33.3	40.0
20	70	9.1	31.3	38.2
21	64.5	8.8	29.5	36.6
22	60.0	8.4	27.6	34.8
23	54.8	8.1	26.1	33.5
24	50.5	7.7	24.3	31.9
25	46.0	7.4	22.9	30.6
26	41.2	7.1	21.4	29.3
27	37.0	6.8	20.1	28.1
28	33	6.5	18.8	26.9
29	29.2	6.2	17.6	25.8
30	25.7	5.9	16.4	24.8
31	22.5	5.6	15.3	23.8
32	19.2	5.4	14.4	23.0
33	16.5	5.1	13.4	22.0
34	13.9	4.9	12.6	21.3
35	11.8	4.6	11.6	20.4
36	9.6	4.4	10.9	19.8
37	8.0	4.1	10.0	19.0
38	6.5	3.8	9.1	18.2
39	5	3.6	8.5	17.7
40	4	3.3	7.6	16.8
41	3.0	3.0	6.8	16.1
42	2.2	2.8	6.2	15.6
43	1.6	2.5	5.5	15.0
44	1.0	2.3	5.0	14.5
45	0.7	2.0	4.3	13.9
46	0.4	1.7	3.6	13.2
47	0.1	1.5	3.1	12.8
48	0.0S	1.3	2.6	12.3
49	0.02	1.0	2.0	11.8

expectancy and age he computed per cent of remainder service life by the use of the expression:

$$\text{Per Cent Remainder Service Life} = \frac{\text{Expectancy}}{\text{Age plus Expectancy}}$$

This expression gives the ratio of expectancy of future service to total service. By applying this per cent to the wearing value of a unit of property and then adding that quantity to the

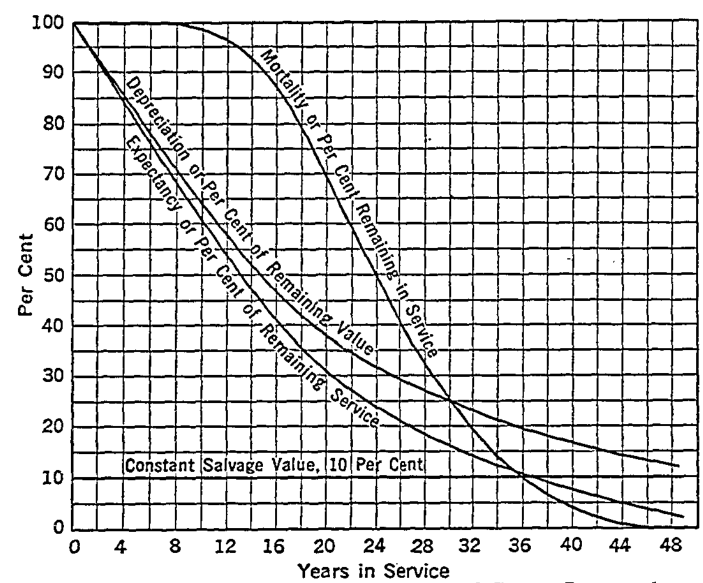


Figure 20. Kates' Depreciation Studies of Steam Locomotives

salvage value, Mr. Kates obtained the per cent remainder value of the unit of property. The curves showing the per cent remaining service and per cent remaining value for locomotives are included in Figure 20 already referred to.

The author considers Mr. Kates' contribution of great value in further establishing the mortality or actuary method.

1921.—In 1921 the author published an article entitled, "Studies in Life of Equipment." [12] Here were presented three

[12] *Engineering News-Record*, December 15, 1918.

mortality tables giving life experience of wooden poles. One of these tables summarized the life experience of the 248,707 wooden poles compiled by Mr. Christiani in 1905 and already discussed in this chapter. The second table gave the life

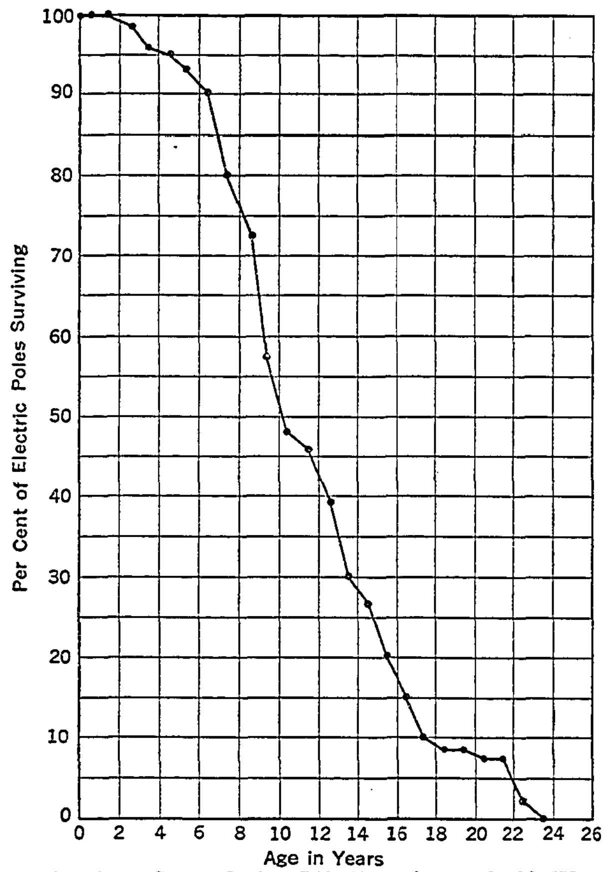


Figure 21. Survivor Curve Giving Life Experience of 309 Wooden Poles of Milwaukee Electric Railway and Light Co.

experience of 1,372 wooden poles compiled by the Electric Company of Missouri, also previously referred to. The third table gave the mortality experience of 309 wooden electric poles of the Milwaukee Electric Railway and Light Co. These latter poles were set in dirt. Replacements were due to butt rot, bad condition, accidents, too short, and civic improvements. These data are shown in Figure 21.

In this article the author pointed out the differences resulting in life estimates, made by use of the various statistical quantities such as mode, median, curtate expectancy, and complete expectancy. The values of these quantities for the three mortality groups are given in Table 11.

Table 11. Life Estimates of Wooden Poles

	309 Poles of M. E. R. & L. Co.	1,372 Poles of Elec. Co. of Mo.	248,707 Poles Compiled by Christiani
Mode	10.0	8.0	8.0
Median	10.0	9.0	10.0
Curtate expectancy	9.93	8.8	10.5
Complete expectancy	10.43	9.3	11.0

Regarding these methods of estimating life, it is obvious that the mode and the median cannot be of much value nor possess much significance as far as the whole group is concerned, because no account is taken of the units as a whole. The mode merely concerns itself with the year during which replacements are a maximum and pays no regard to the general distribution during the other years. The median likewise is determined wholly by the 50% point on the mortality curve and also fails to take into account the distribution before and after that time. The "life expectancy" is a much more accurate means of representing the average life of a given type of unit. By this term is meant the average number of years which units of a given age will survive. To determine the average after-life find the sum of the products of age multiplied by the number of units removed during that year. This sum represents the total unit years lived by the 100 units. The average number of years each unit can therefore be expected to live is this sum divided by 100.

1924—Great Northern Railway.—In 1924, the Great Northern Railway presented certain testimony before the Interstate Commerce Commission relative to Valuation Docket No. 327, covering the life experience of ties and rolling stock.

For some time previous to 1924, there had been established on this railroad a number of sections of track approximately one mile in length for the purpose of studying cross tie life. On these sections a record was taken at the time of construction of the track of the actual number and kind of ties in the section. The ties were all marked so that they could be identified. Since the installation of the ties a record has been kept by annual inspections of the number of ties remaining in the track and the age of the ties retired during the preceding year. These "tie life miles," as they were called, were scattered over the line from North Dakota to Washington. In 1924 the complete life history of the ties originally placed was available for nine of the sections. The record of retirements for these sections is shown in Table 12 and the corresponding survivor curves are given in Figure 22. On this chart the survivors are plotted against age, expressed in per cent of average life. Regarding this exhibit (No. 78) Henry K. Dougan testified as follows:

"As for instance, we find considerable variation between the time of retirement of the first and last ties. We also find that the average life varies with different lines, but we find also that while the average life varies as between different test sections, on all test sections the ties begin to be retired just about half of their average life, and the last of the retirements on all of the sections is at about 150% of the average life.

"Looking at the diagram you will note that while there is some variation between the curves of the different test sections, they generally follow the same direction and begin and end at approximately the same place."

The right-hand column in Table 12 shows the total for all ties in the nine sections. The curve representing this summary

Table 12. Life of Ties as Determined from Tie Life Mile Records

(History of ties remaining in track at each annual inspection of certain miles which are designated as "tie life miles")

Age of Ties Years	NUMBER OF ORIGINAL TIES REMAINING IN TRACK									Total
	Kalispell Division			Spokane Division		Butte Division				
	M.P. 1217–M.P. 1218 near Whitefish Placed 1904	M.P. 1213–M.P. 1214 near Col. Falls Placed 1904	M.P. 1263–Br. 116 near Fortine Placed 1904	M.P. 1374–M.P. 1375 near Moravia Placed 1904	Br. 249–I.PE. 1430 near Newport Placed 1908	M.P. 70–M.P. 71 near Conrad Placed 1902	M.P. 10–M.P. 11 near Rimrock Placed 1908	M.P. 202–M.P. 203 near Wayne Placed 1911	M.P. 208–M.P. 209 near Swift Placed 1911	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0........	2,708	2,823	2,168	2,918	2,894	2,601	2,916	1,340	850	21,218
1........	2,708	2,823	2,168	2,918	2,894	2,601	2,916	1,340	850	21,218
2........	2,708	2,823	2,168	2,918	2,894	2,601	2,916	1,340	850	21,218
3........	2,708	2,823	2,168	2,918	2,894	2,601	2,916	1,340	850	21,218
4........	2,708	2,823	2,168	2,918	2,739	2,601	2,897	1,340	850	21,044
5........	2,708	2,823	2,060	2,914	2,237	2,601	2,572	1,293	850	20,058
6........	2,343	2,493	1,538	2,382	1,394	2,601	2,378	944	835	16,908
7........	2,211	2,355	1,150	1,996	958	2,601	1,418	768	453	13,910
8........	1,778	2,061	265	1,167	514	2,437	1,121	269	149	9,761
9........	1,410	1,169	49	612	22	1,928	471	113	98	5,872
10........	959	940	0	441	0	1,765	0	0	0	4,105
11........	809	760		353		1,461				3,383
12........	647	445		230		952				2,274
13........	464	322		111		642				1,539
14........	118	0		0		365				483
15........	30					362				392
16........	0					119				119
17........						79				79
18........						35				35
19........						0				0
Average Life.....	9.48	9.24	6.83	8.00	6.22	11.40	7.22	7.03	7.31	

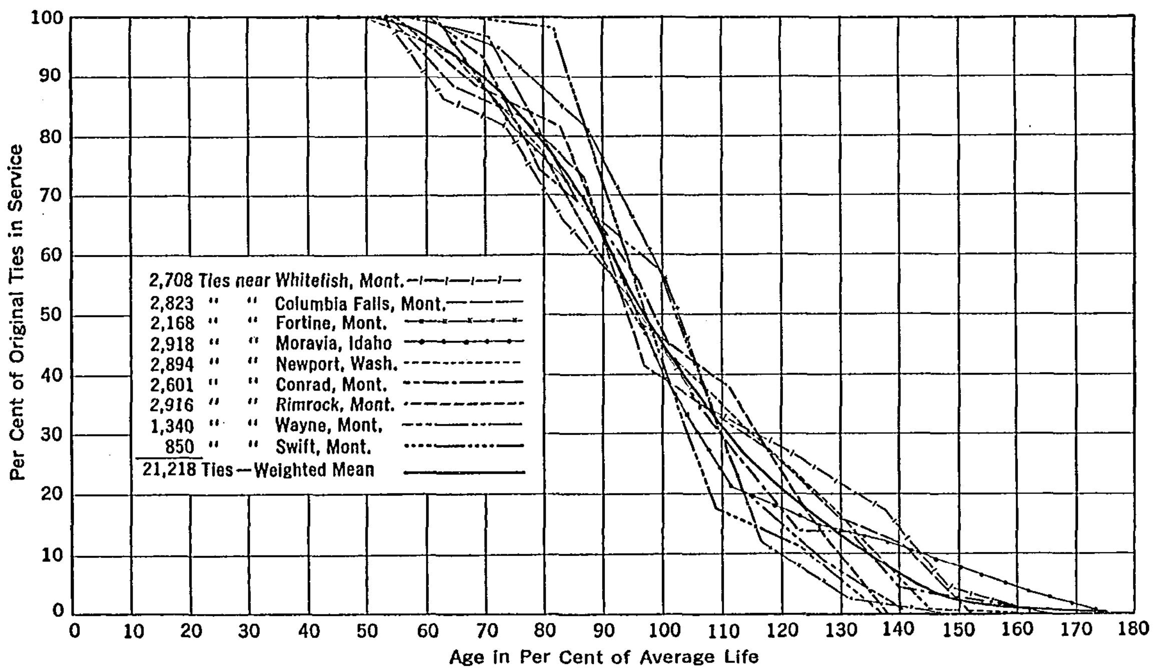


Figure 22. Tie Life History Curves of Various Test Miles

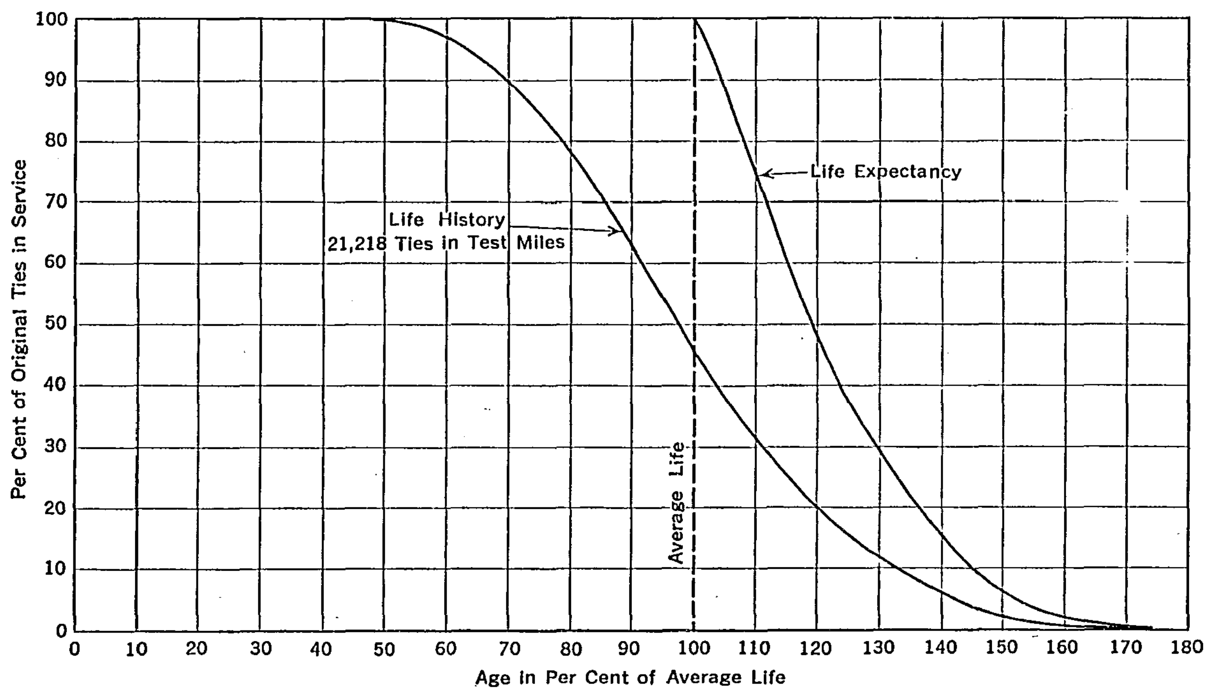


Figure 23. Tie Life History and Expectancy Curves

together with the expectancy curve for this average line is shown herewith as Figure 23.

In addition to the tie life experience obtained from the nine track sections, Mr. Dougan also introduced testimony showing the tie life experience on four lines of the Great Northern Railway System built between 1901 and 1907. These data are shown in Figure 24. As shown on the chart these curves summarize the life experience of a total of 746,599 ties. Mr. Dougan gave the following explanation regarding the method of securing the life history of these ties on the four lines:

"For these lines we have to start with the record of the number of ties placed on construction. This record was taken in each case from the records of the engineering department which are made up at the time of the construction of the line, and show the number of ties in track according to actual count. Further, we investigated the superintendent's material distribution to find the number of ties placed in renewals, and we made this estimation, that for these lines the renewals equalled the ties which were retired. This is not exactly true for it sometimes happens that one large tie is taken out of the track and two smaller ties are put in to replace it. Or, which happens probably less frequently, one large tie might replace two smaller ties. In taking the number of ties we paid no attention to the ties that were put in for additions and betterments. However, we picked these four lines, because the additions and betterments were not heavy, especially during the first years of their life, and therefore the additions and betterments would not influence our data to any great extent. We show here information not only by years but by months, because the superintendent's monthly distribution of material is made up monthly and we could therefore get to the number of ties placed in track each month.

"For the Berthold-Crosby line there had not been enough renewals since construction to account for all of the ties in

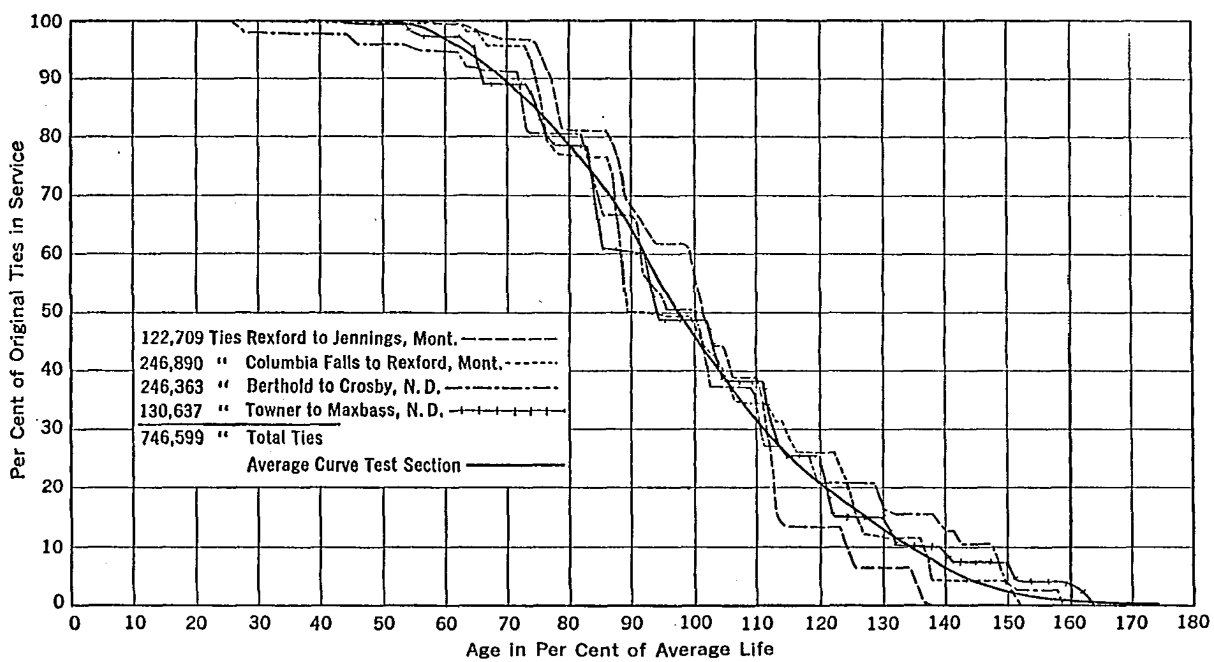


Figure 24. Tie Life History Curves of Various Lines

track. The discrepancy is a little less than 3%. We assumed this 3% of ties was replaced one month after our last date.

"There is another thing that we did not take into consideration in these data, and that is that some of the ties placed and charged to replacement, would themselves probably replace other ties that were placed in replacement. We considered replacement ties as replacing only the original ones. I do not think this would greatly influence the study. I want to explain this because the data are not exact nor susceptible of exact treatment as are the data obtained from the tie life miles, despite the fact that we have a history here dealing with nearly three quarters of a million ties.

"We worked up tables then deducting from the original ties the number of ties placed each month, charged to replacement, regardless of the fact that some of those ties might have replaced some few ties placed in additions and betterments or replaced replacement ties."

There is also plotted for comparison on Figure 24 as a heavier line the average tie life history curve for the nine test sections, already shown in Figure 22. The agreement between this average curve and the experience on the four lines is quite apparent. In fact, Mr. Dougan claimed that there is closer agreement than there should be when he said:

"In fact there is a greater coincidence than there really should be, because the curve of the various lines should flatten off a little bit more at the ends than it does. It does flatten more than the average curve of the tie life miles, but it would flatten more if we had taken into consideration the number of renewals of renewal ties."

Life Experience of Rolling Stock.—In addition to this mortality experience of ties, testimony was also introduced giving the life experience of rolling stock. This was introduced as Exhibit No. 71 and included life experience of box cars of 28,000, 40,000, 60,000 and 80,000 lb. capacity; freight train cars, such as stock cars, flat cars, ore cars, both wood and steel, and refrigerator cars; passenger train cars including coaches, sleepers, baggage cars, mail cars, express cars, and express refrigerator cars; work equipment cars including Roger ballast cars and sand cars. Only two of these life studies were entirely complete as to life. These pertain to box cars (28,000 lb.) and ore cars (wood). They are shown herewith as Figures 25

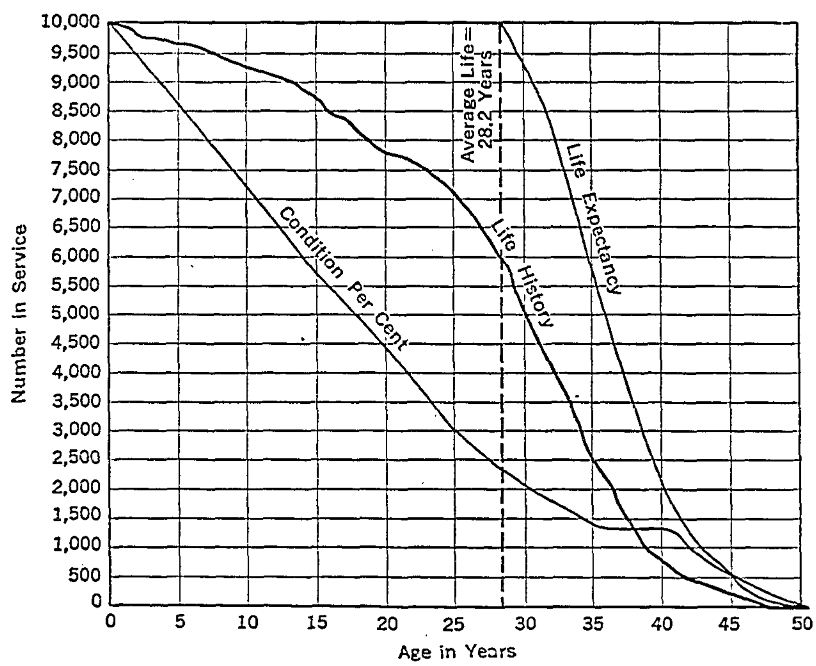


Figure 25. Life History and Expectancy Curves of Box Cars (28,000 lb. capacity)

and 26 respectively. As an example of the uncompleted life experience, Figure 27, on stock cars, is included.

All of these life studies were made by following through the history of practically every car purchased by the Great Northern Railway or its predecessors prior to the year 1906, and many which were purchased after that date. The retirements were undoubtedly affected by all the changing operating conditions since the first car was purchased in 1869.

The above described mortality studies made by the Great Northern Railway constitute another colossal contribution to the life experience statistics of physical property. These studies are being continued and additional data will become available in the future.

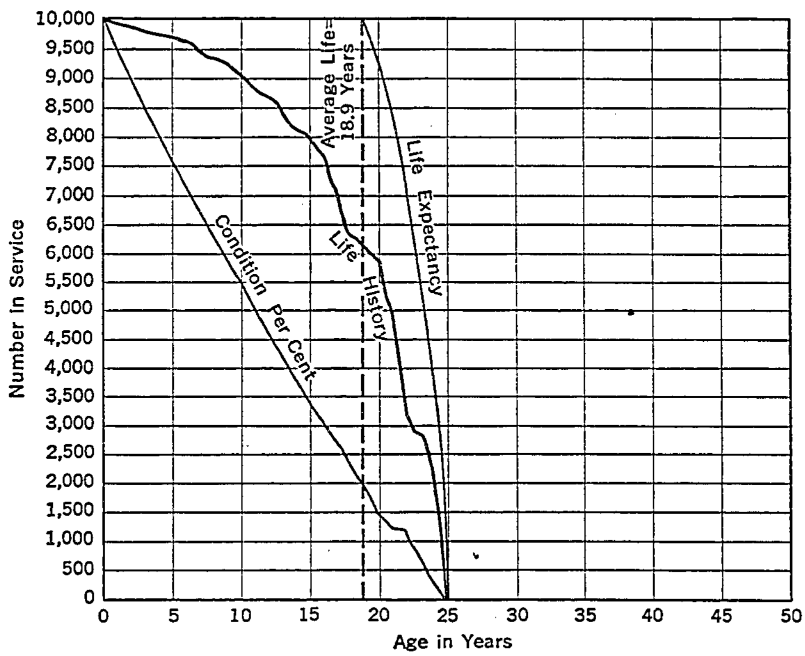


Figure 26. Life History and Expectancy Curves of Wooden Ore Cars

1925—**Lew Wallace.**—At the annual meeting of the American Society of Agricultural Engineers held in 1925 Lew Wallace presented a paper entitled "Depreciation of Farm Machinery." This paper was a report of a survey made by Mr. Wallace in Iowa under the auspices of the Engineering Experiment Station of Iowa State College. In the survey, data were collected of life experience of various kinds of machinery, cost of repairs, etc. Expressions of weaknesses in design and construction were also obtained. Much informa-

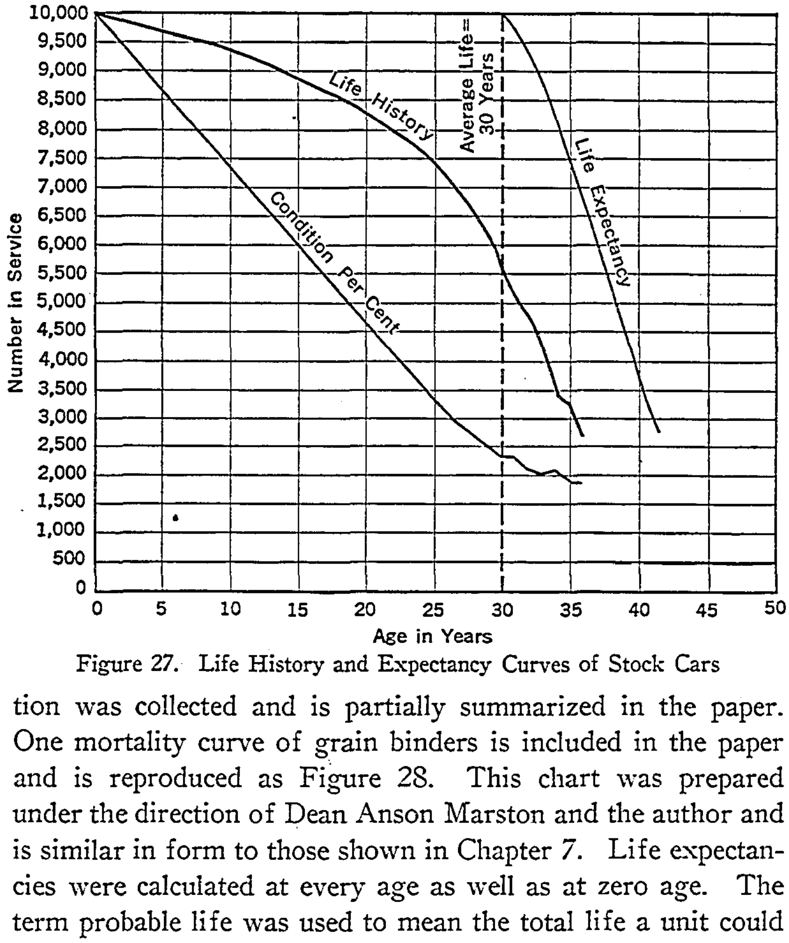


Figure 27. Life History and Expectancy Curves of Stock Cars

tion was collected and is partially summarized in the paper. One mortality curve of grain binders is included in the paper and is reproduced as Figure 28. This chart was prepared under the direction of Dean Anson Marston and the author and is similar in form to those shown in Chapter 7. Life expectancies were calculated at every age as well as at zero age. The term probable life was used to mean the total life a unit could be expected to live at a given age. Probable life was, then, equal to the "already expired" life plus "unexpired" life or expectancy. Since expired life is equal to age, probable life can be defined as age plus expectancy. By noting the manner in which probable life is plotted it will be evident that all three quantities, age, expectancy, and probable life, can be read on the age scale.

This method of presentation is further discussed in Chapter 7.

As far as the author is aware, the mortality curve shown above on grain binders is the first mortality experience that has been compiled of agricultural machinery. It is therefore the first contribution from an almost limitless field.

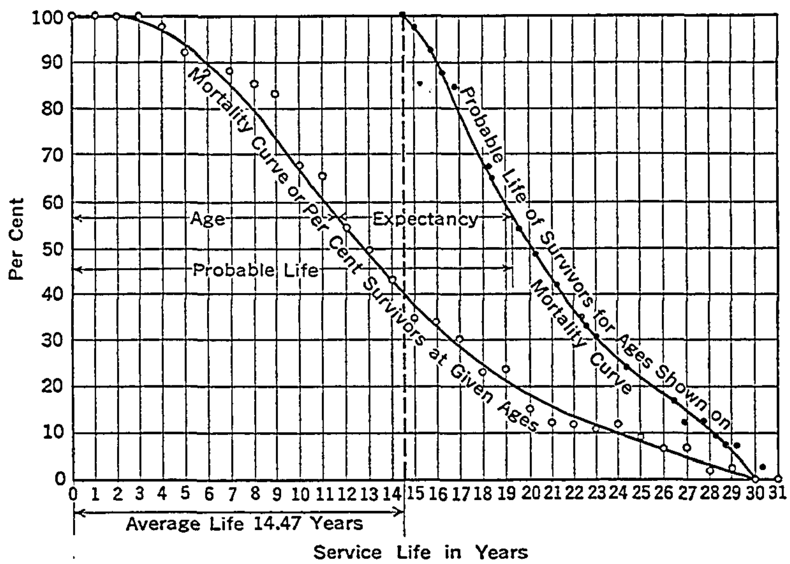


Figure 28. Survivor and Probable Life Curves of Grain Binders as Found by the Engineering Experiment Station of Iowa State College

1926—Robley E. Winfrey.—The first comprehensive study of the life of automobiles was made in 1926 by Robley E. Winfrey, then a graduate student at Iowa State College. The results of this study appear in his Master's thesis entitled, "Registration and Mortality Studies of Iowa Automobiles." By a study of Iowa registration figures on automobiles for 1922 and 1924, and by reference to records showing those cars dismantled, 1922 to 1925, mortality curves were constructed showing the number of vehicles surviving at any age. Separate mortality curves for Ford cars, and other makes, were constructed as shown in Figure 29.

Regarding these curves, Mr. Winfrey commented as follows: "Strictly speaking, the automobile mortality curves given here are not true mortality curves as they are not based on the same set of original units. It should also be noted that the units comprising the mortality curves did not possess the

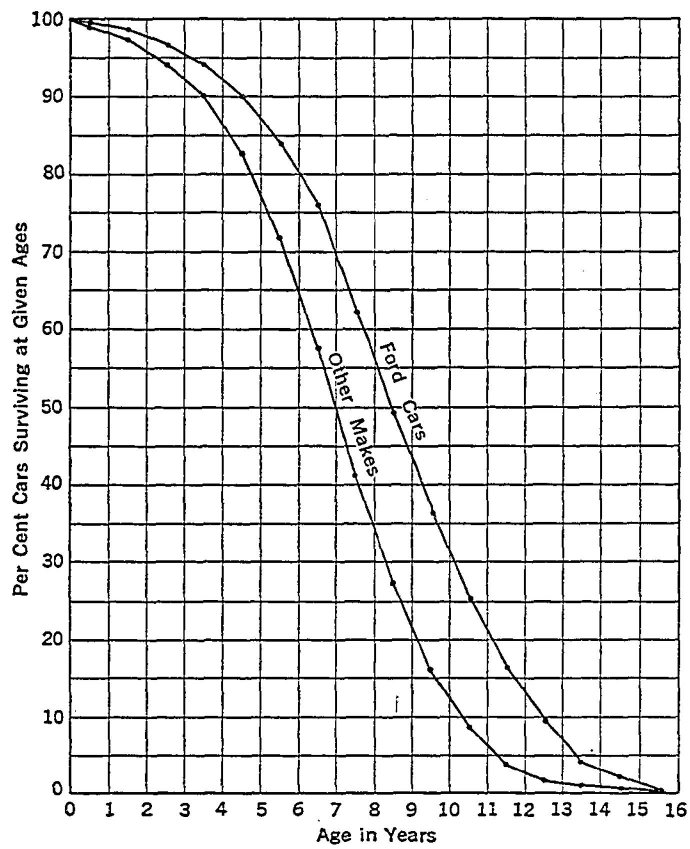


Figure 29. Experience in Iowa with Life of Automobiles

same amount and quality of potential service as they were mechanically different. The cars manufactured in 1909, 1910, and 1911 do not have the same average life as the cars manufactured in 1922, 1923, and 1924, though they were given the same weight in determining the mortality curve. . . .

The units differ also in the respect that the annual mileage for the older automobiles is less than that of the newer models. The type of roads has changed materially so that the treatment accorded the cars is unequal from the standpoint of effect of road conditions."

From these curves the average life and expectancy at various ages were determined.

1928—American Telephone and Telegraph Company.— In 1928, the American Telephone and Telegraph Company presented testimony before the Interstate Commerce Commission in the matter of depreciation charges of telephone companies, Docket No. 14,700. The testimony of two of the four witnesses was devoted almost entirely to the defense of the actuarial method in the determination of average lives and depreciation reserves. As already noted, one of the Bell System companies, the New York Telephone Company, was making studies of life experience of telephone equipment as early as 1916. The only mortality data introduced at this time comprised four summaries of historic data referred to as Charts Nos. 9, 10, 11 and 12. Chart No. 9 is reproduced herewith as Table 13 to show the method employed in constructing the table. The charts relate to aerial cable, exchange underground cable, toll poles, and non-multiple private branch exchange equipment respectively. Regarding these charts D. R. Belcher testified in part as follows:

"During the past three or four years we have received from the operating telephone companies several hundreds of summaries of the form illustrated by Charts 9, 10, 11 and 12 (see Table 13), and covering in the aggregate the six classes of plant under Group 4. These summaries have formed the basis of extensive study and analysis on our part.

"Each of these summaries shows for a given class of telephone property under investigation how many property units, expressed in dollars upon a uniform cost level, were placed in service during each calendar year, and how many of these were retired from service during the same and each subsequent calendar year.

"There is a definite analogy between the types of data analyzed by the life insurance actuary and the type of data which we have as to the retirement experience of telephone property comprised in Group 4.

"As Mr. Woodford has pointed out in his discussion of Charts 9, 10, 11 and 12 (Table 13), we have information, first, as to the amount of plant in service at each year of age, and, second, as to the amount of plant which was actually retired during each year of age.

"Stated in the language of the actuary, we know for any given year of age the amount of plant 'exposed to the risk of dying' and the amount which actually did 'die.' The ratio between the amount dying and amount exposed to the risk of dying constitutes the mortality rate for that year of age.

"Referring, for example, to the totals shown at the bottom of the page (Table 13), we find in column 10 that at the beginning of the tenth calendar year following the year of placing, $1,790,199 of telephone plant was in service and that, of this amount, $109,794 of plant reached the end of its service life during the same year and was retired. We say, then, that the mortality rate for all the plant appearing in column 10 is 109,794 divided by 1,790,199, or 6.1%.

"Starting back at the upper left-hand corner of this table, it appears that during the calendar year 1902 an amount of $29,654 of plant was placed in service, and that since no retirements from that plant are shown in column 0 the entire $29,654 of plant survived the calendar year of placing and was in service at the beginning of the following year as shown by the upper figure in column 1.

"Since the placing of the $29,654 of plant was distributed throughout the calendar year 1902, this plant, on the average, may be said to have attained an age of ½ year at the beginning

Table 13. Summary of Historical Data for Mortality Study of Exchange Aerial Cable

(Figures represent equated property units expressed in dollars.)

YEAR OF PLACING	(a) TOTAL AMOUNT OF PLANT PLACED	(b) TOTAL AMOUNT OF PLANT RETIRED	(c) TOTAL AMOUNT OF PLANT STILL IN SERVICE (a)-(b)	UPPER FIGURES: PLANT REMAINING IN SERVICE AT BEGINNING OF Nth CALENDAR YEAR AFTER YEAR OF PLACING LOWER FIGURES: PLANT RETIRED DURING Nth CALENDAR YEAR AFTER YEAR OF PLACING																									
				N = 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
				xxxx																									
				xxxx																									
				xxxx																									
				xxxx																									
				xxxx																									
				xxxx																									
1902	29,654	29,287	367	xxxx -	29,654 194	29,460 2,262	27,198 62	27,136 -	27,136 -	27,136 542	26,594 2,238	24,356 2,621	21,735 1,793	19,942 327	19,615 2,023	17,592 4,348	13,244 4,650	8,594 3,342	5,252 1,071	4,181 -	4,181 -	4,181 1,030	3,151 1,665	1,486 -	1,486 1,119	367 -	367 -	367	
1903	41,509	38,085	3,424	xxxx -	41,509 -	41,509 -	41,509 427	41,082 1,426	39,656 3,746	35,910 2,013	33,897 -	33,897 3,438	30,459 214	30,245 2,346	27,899 9,178	18,721 5,706	13,015 1,153	11,862 2,501	9,361 -	9,361 695	8,666 668	7,998 710	7,288 2,688	4,600 1,025	3,575 -	3,575 45	3,530 165		
1904	80,821	67,086	13,735	xxxx 1,303	79,518 252	79,266 2,220	77,046 3,434	73,612 1,790	71,822 1,728	70,094 7,185	62,909 1,900	61,009 3,581	57,428 4,292	53,136 13,085	40,051 3,309	36,742 3,264	33,478 4,855	28,623 4,004	24,619 251	24,368 791	23,577 876	22,701 4,844	17,857 1,946	15,911 922	14,989 616	14,373 638			
1905	198,581	176,510	22,071	xxxx -	198,581 -	198,581 1,285	197,296 1,724	195,572 2,045	193,527 11,523	182,004 4,387	177,617 19,108	158,529 12,267	146,262 13,613	132,649 12,803	119,846 28,096	91,750 11,398	80,352 10,037	70,315 44	70,271 2,595	67,676 2,492	65,184 912	64,272 10,399	53,873 2,077	51,796 2,861	48,935 26,864				
1906	178,877	145,907	32,970	xxxx 369	178,508 5,756	172,752 -	172,752 368	172,384 989	171,395 6,886	164,509 3,851	160,658 23,161	137,497 20,633	116,864 6,739	110,125 10,697	99,428 12,476	86,952 17,196	69,756 4,552	65,204 3,602	61,602 5,875	55,727 2,992	52,735 3,489	49,246 4,164	45,082 2,981	42,101 9,131					
1907	118,070	83,620	34,450	xxxx -	118,070 637	117,433 1,125	116,308 2,727	113,581 2,991	110,590 2,885	107,705 4,574	103,131 6,817	96,314 2,582	93,732 10,959	82,973 10,318	72,655 5,492	67,163 5,569	61,594 3,842	57,752 4,580	53,172 4,028	49,144 3,092	46,052 1,352	44,690 7,155	37,539 3,659						
1908	52,653	43,841	8,812	xxxx -	52,653 -	52,653 435	52,218 -	52,218 468	51,750 6,780	44,970 6,472	38,498 4,131	34,367 4,178	30,189 7,957	22,232 -	22,232 261	21,971 867	21,104 1,112	19,992 4,340	15,652 2,401	13,251 2,677	10,574 1,327	9,247 434							
1909	89,856	77,089	12,767	xxxx 225	89,631 -	89,631 625	89,006 2,466	86,540 1,302	85,238 6,376	78,862 3,372	75,490 8,975	66,515 8,109	58,406 1,838	56,568 10,873	45,695 403	45,492 11,430	34,062 13,156	20,906 3,564	17,342 1,341	16,001 2,699	13,302 535								
1910	159,651	109,581	50,070	xxxx -	159,651 174	159,477 1,071	158,406 10,263	148,143 17,047	131,096 9,846	121,250 20,788	100,462 8,352	92,110 2,158	89,952 3,737	86,215 3,784	82,431 6,245	76,186 7,593	68,593 8,294	60,299 672	59,627 3,184	56,443 6,373									
1911	162,576	76,442	86,134	xxxx 391	162,185 464	161,721 5,470	156,251 6,791	149,460 10,844	138,616 6,311	132,305 5,202	127,103 2,132	124,971 579	124,392 4,890	119,502 6,450	113,052 6,625	106,427 2,584	103,843 6,928	96,915 2,692	96,223 10,089										
1912	243,731	85,978	157,753	xxxx 530	243,201 3,729	239,472 -	239,472 4,969	234,503 7,037	227,466 290	227,176 11,418	215,758 4,237	211,521 6,409	205,112 3,939	201,173 16,933	184,240 4,658	179,582 1,436	178,146 5,052	173,094 15,341											
1913	254,079	97,680	156,399	xxxx 20	254,059 3,470	250,589 614	249,975 7,158	242,817 4,188	238,629 5,094	233,535 3,104	230,431 1,305	229,126 19,284	209,842 8,907	200,935 10,704	190,231 5,516	184,715 7,771	176,944 20,565												
1914	314,832	72,006	242,826	xxxx 511	314,321 683	313,638 1,961	311,677 8,586	303,091 4,746	298,345 490	297,855 5,003	292,852 10,722	282,130 12,225	267,905 2,331	267,574 3,557	264,017 7,028	256,989 14,163													
1915	158,586	31,229	127,357	xxxx 217	158,369 -	158,369 -	158,369 -	158,369 3,528	154,841 2,266	152,575 8,550	144,025 12,451	131,574 258	131,316 331	130,985 987	129,998 2,641														
1916	300,713	29,792	270,921	xxxx 183	300,524 -	300,524 1,235	299,289 20	299,269 1,709	297,560 1,516	296,044 5,889	290,155 5,987	284,168 3,310	280,858 2,913	277,947 7,026															
1917	219,972	14,552	205,420	xxxx 166	219,806 1,022	218,784 1,829	216,955 299	216,656 923	215,733 167	215,566 801	214,765 2,778	211,987 3,762	208,225 2,805																
1918	142,706	24,511	118,195	xxxx 314	142,392 -	142,392 1,270	141,122 3,277	137,845 7,277	130,568 79	130,489 676	129,813 3,866	125,947 7,752																	
1919	84,535	5,518	79,017	xxxx 144	84,391 -	84,391 1,304	83,087 118	82,969 551	82,418 191	82,227 1,750	80,477 1,460																		
1920	90,837	3,681	87,156	xxxx 293	90,544 567	89,977 264	89,713 368	89,345 124	89,221 1,481	87,740 584																			
1921	211,728	8,802	202,926	xxxx 268	211,460 179	211,281 1,232	210,049 964	209,085 1,609	207,476 4,550																				
1922	307,899	4,925	302,974	xxxx 66	307,833 711	307,122 1,409	305,713 1,829	303,884 910																					
1923	242,009	8,709	233,300	xxxx 1,009	240,995 144	240,851 2,035	238,816 5,516																						
1924	155,232	790	154,442	xxxx 480	154,752 28	154,724 282																							
1925	830,051	1,435	828,616	xxxx 90	829,961 1,345																								
1926	252,988	1,337	251,651	xxxx 1,337																									
TOTAL	3,922,194	1,236,403	2,685,791	xxxx 7,972	3,662,571 19,341	3,516,614 27,930	3,452,242 59,353	3,353,559 73,502	2,961,085 79,205	2,687,952 96,101	2,504,657 119,632	2,306,018 112,946	2,074,877 79,258	1,790,199 103,794	1,407,484 93,865	1,188,262 91,307	854,129 82,074	615,656 44,763	413,120 32,839	294,152 19,831	224,251 9,169	202,335 28,817	164,706 14,360	115,896 13,941	68,965 28,599	18,315 683	3,897 105	367	

of the year 1903. The $194 of plant shown as retired in column 1 was accordingly retired between the ages of ½ year and 1½ years.

"Hence, by definition, the mortality rate computed out of this rectangle or out of any aggregate of rectangles in column 1 provides a mortality rate for age ½ year. A similar interpretation must be placed on mortality rates derived from the succeeding columns, and the mortality rate of 6.1%, which I discussed a moment ago, as determined from column 10, is to be interpreted as the mortality rate for age 9½ years.

"So far as the early ages are concerned, a very large amount of data is thus afforded for the determination of mortality rates. However, it is not at all certain that the retirement experience of some years ago is sufficiently similar to that of recent years to warrant the determination of mortality rates out of the combined data. From this point of view it would seem preferable to limit our attention to the experience of, say, only the final year, 1926. For example, by observing the data included only in the final rectangle of each of these columns, that is, a diagonal array of rectangles running from the lower left-hand corner to the upper right-hand corner, we could determine a mortality rate for each age there represented.

"Since each of the upper figures in this diagonal array represents plant in service at the beginning of 1926 and each of the lower figures represents plant retired during 1926, it follows that the series of mortality rates so determined would represent the retirement experience of the year 1926 taken by itself.

"This diagonal array may advantageously be somewhat broadened and we may study, for example, all of the plant in service and the plant retired during the three-year period 1924, 1925, and 1926.

"These figures, for purpose of illustration, have been enclosed in a heavy zig-zag outline on Table 13. By totaling for each age the dollars of plant in service and also the dollars of plant retired, we calculate a series of mortality rates for this three-year band.

"Because of the large volume of data included and also the fact that any unusual behavior within a single calendar year will be somewhat minimized, it is obvious that such a band provides a more satisfactory basis of analysis than would the figures of a single year. We have ordinarily followed this procedure in our analyses.

"Having arrived at mortality rates for a particular band of years, our next problem is to determine from these mortality rates the average service life which they represent. As I have already pointed out, the established actuarial approach to such a problem is to calculate from these mortality rates a series of figures which constitute what is known as a 'life table.'

"We have adopted an identical procedure, and the resulting table represents the future life experience of any initial installation of plant which is subject to these particular mortality rates. For example, if $1,000,000 of plant was installed in a given year, we may apply the mortality rate which has been calculated for the year of installation to find out how much will survive to the end of that year, thus attaining an average age of ½ year. We may apply to the latter figure the mortality rate for age ½ year to find out how much will survive to age 1½ years. To the latter figure we apply the mortality rate for age 1½ years to determine how much will survive to age 2½ years. This process is continued until the original $1,000,000 of plant has entirely disappeared, or at least until all of our available mortality rates have been employed. A life table so determined is usually referred to as an 'observed life table,' to indicate that no process of smoothing or graduation has yet been applied.

"In order to illustrate observed life tables developed from telephone property experience, I have calculated one life table for each of the summaries of mortality data shown in Charts 9, 10, 11, and 12. In each case, for the purpose of the present illustration, I have limited the calculation to the final three-year band of data. The resulting four life tables are shown graphically in Figure 30.

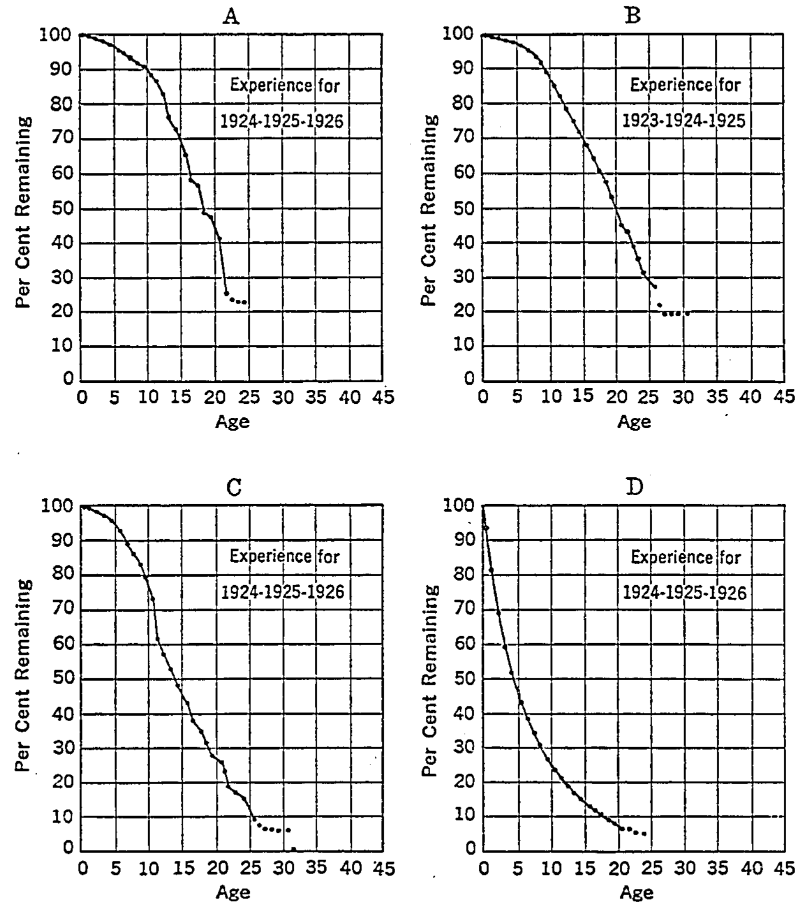


Figure 30. Observed Life Charts (Tables). (Derived from "Final Summary of Historical Data for Mortality Study." See Table 13.)

"Life table A has been calculated, as described, directly from the mortality data shown in the final three-year diagonal band of Chart 9 (Table 13). Starting out with 100% at age 0, we find that the amount remaining in service each year thereafter is reduced only gradually in the early years, so that at age 5½ years some 96% of the original installation still remains. As the age advances, retirement proceeds for a time at a somewhat higher rate, the per cent remaining curve drops more rapidly, and finally at age 24½ years we have left only 23% of the original installation. It happens that this particular band of experience contains no data as to plant retirements at ages beyond 24½ years.

"Life tables B, C, and D have been similarly calculated. Life table D is a good illustration of a case in which the amount remaining in service is reduced rapidly in the early years, so that at age 5½ years only about 45% of the original installation remains."

The foregoing testimony clearly shows the tremendous advances in the art and use of mortality studies of physical property in modern business since the year 1903. The advances made to date are, however, but a mere beginning and much further progress can be confidently looked forward to.

CHAPTER 2

METHODS OF COMPILING MORTALITY TABLES

Individual Unit Method.—The simplest way of compiling life experience data of a given type of physical property is to record the age in years of each individual unit of property of that class as it goes out of service. When a large number of such individual lives have thus been recorded the data can be summarized and presented as shown in Table 14. This table then shows how many units of property of a given class are taken out after various years of service. Thus, the table shows that 5 units were removed at age 1, 12 at age 2, 18 at age 3, 20 at age 4, etc., until the end. When the data of this table are plotted as shown in Figure 31, the well-known "distribution" or "frequency" curve results.

To obtain the survivor curve it is only necessary to determine the total number of units in the group, and on the assumption that they were all placed in service at age 0 determine how many survive the subsequent ages. Using the data in column (2) of Table 14, it will be seen that there are 143 units in the group. It will be assumed that they were all placed in service at age 0, which is really true except that age 0 may not be the same calendar time for all of the units. Of these 143 units, 5 units acquired an age of only 1 year when they were removed from service, leaving 138 as the number of units surviving the first year. Likewise, 12 units reached an age of only 2 years when they were removed from service, leaving 126 as the number of units surviving the second year. The calculations are continued in this manner to the end of the table with the resulting values as shown in column (3) of the

Table 14. Life Experience Data of Physical Property

Age (Years) (1)	Number of Units Removed During ½ Year Preceding and ½ Year Following Given Ages (2)	Number of Units Remaining in Service at the Given Ages (3)
1..........	5	143
2..........	12	138
3..........	18	126
4..........	20	108
5..........	24	88
6..........	19	64
7..........	17	45
8..........	14	28
9..........	9	14
10..........	5	5
	143	

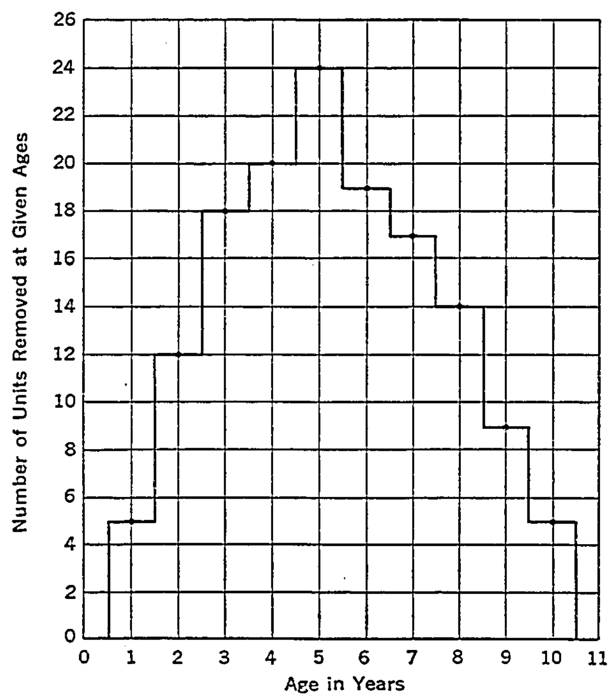


Figure 31. Distribution Curve of Data in Table 14

table. Such a table showing the number of units of a given group removed each year as well as the number of units surviving each year constitutes the so-called "mortality table," and the graph obtained from plotting the survivor data therefrom is the commonly called mortality curve. This curve is shown in Figure 32 for the illustrative data given in Table 14.

It should be pointed out that mortality tables prepared as outlined above record the lives of individual units to the nearest year only. Hence, units shown as having been removed at a given age may in reality have been removed any time during the period from ½ year preceding to ½ year following the recorded age of removal. This fact must be given due attention in any plotting of curves and in any calculations of life characteristics as will be shown hereinafter. In the illustration, for example, the 143 units in service at age 0 remain in service up to age ½ year, when some of the 5 units begin to go out of service. These 5 units do not all become removed until age 1½ years, leaving 138 units in service at age 1½ years. Likewise, the next 12 units do not all become removed until age 2½ years, leaving 126 units in service up to an age of 2½ years. This is a fundamental consideration and must be taken into account when data are compiled in this manner. The survivor curve of Figure 32 was plotted on this basis.

Disadvantages of Individual Unit Method.—The principal disadvantage of compiling mortality tables by the method just described is that many years of observation are required. Theoretically, to compile a complete table means recording lives until the longest-lived unit has been replaced. This may be a period of 50 years or more with some classes of property. In such long periods of time operation practices may improve and administration policies affecting maintenance and repairs may change. Tables compiled in this manner may, therefore, lose some of their value due to there being many human factors involved.

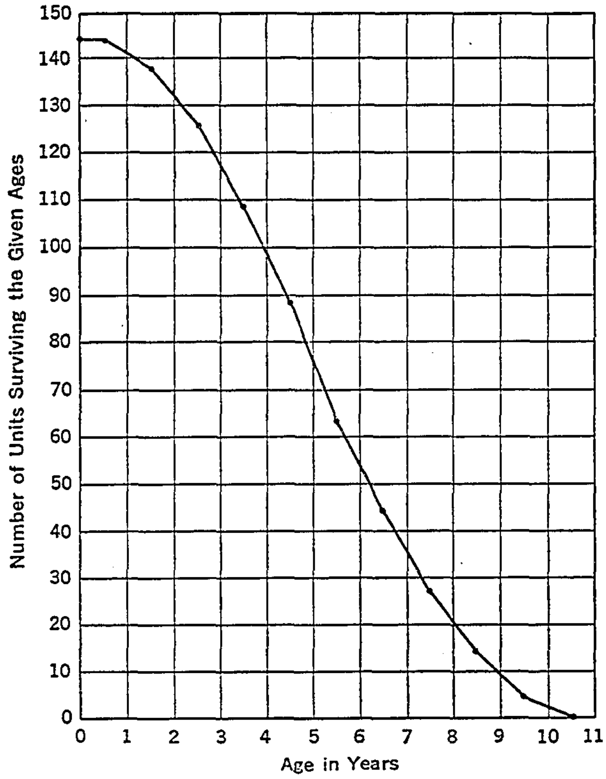


Figure 32. Survivor Curve of Data in Table 14

Annual Rate Method.—Contrasted with the individual unit method is the annual rate method of compiling mortality tables. In this method observations need only be made for a period of years such as 1, 2, 3, 5, or 10 years. The period selected should be a normal period thus giving representative retirement or replacement rates. The ideal period is one so short that it only reflects present policies and standards, and yet long enough so that numerous replacements will have been made at each age of property in existence. The steps in the method are as follows:

1. Determine the number of units of property of a given

class retired each year during the observation period and the age at which retirement occurred. Then determine the *average* number of units of property of each age retired per year for the period of observation. This information will then give the average number of units of property retired between the ages

Table 15. Annual Retirements of Water Stations in a Large Railroad System

Age Interval		Annual Retirements in Dollars					Total Retirements per year 1922–1926 Incl.	Average Retirements per year 1922–1926 Incl.
From	To	1922	1923	1924	1925	1926		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Years	Years							
0	½	$ 100	$ 169			$ 105	$ 374	$ 75
½	1½		1,207	$ 2,721	$ 1,453	473	5,854	1,171
1½	2½	1,401	1,067	1,450	601	3,240	7,759	1,552
2½	3½	40,407	8	2,213	1,807	5,558	49,993	9,998
3½	4½	307	3,575	294	1,281	2,727	8,184	1,637
4½	5½	1,216	309	6,755	2,239	1,500	12,019	2,404
5½	6½		174	535	24,421	16,808	41,938	8,388
6½	7½	87	602	589	3,890	4,612	9,780	1,956
7½	8½	1,694	3,150	1,526	16	305	6,691	1,338
8½	9½	25			1,011	1,829	2,865	573
9½	10½	1,167		2,424	20,094	767	24,452	4,890
10½	11½	2,218	145	5,604	3,663	19,543	31,173	6,235
11½	12½		1,925		14,848	931	17,704	3,341
12½	13½	392	7,321	2,438	2,293	17,803	30,247	6,049
13½	14½	1,718	239	1,765	2,025	7,416	13,163	2,633
14½	15½	1,756	4,968		74	5,352	12,150	2,430
15½	16½	550	609	1,591	4,773	224	7,747	1,549
16½	17½	416		182	13,299	2,287	16,184	3,237
17½	18½			4,473	4,005		8,478	1,695
18½	19½	14,871	1,743	1,388	1,140	28,378	47,520	9,504
19½	20½	2,813	1,045	49,380	1,431	8,216	62,885	12,577
20½	21½	2,046	1,981	3,803	2,826	1,194	11,850	2,370
21½	22½	4,598	3,820	16,154			24,572	4,914
22½	23½	3,515	1,002	10,694	18,561	2,473	36,245	7,249
23½	24½		1,123		12,729	2,979	16,831	3,366
24½	25½	1,818		1,137	1,630	7,917	12,502	2,500
25½	26½				1,773	701	2,474	495
26½	27½		1,668			5,214	6,882	1,376
27½	28½	413		3,919		185	4,517	902
28½	29½		558		3,189		3,747	749
29½	30½							
30½	31½			1,587			1,587	317
31½	32½	1,638				2,181	3,819	764
32½	33½	7,047			1,516		8,563	1,713
33½	34½		2,409				2,409	482
34½	35½							
35½	36½					3,563	3,563	713
36½	37½				3,001		3,001	600
37½	38½	1,782				1,913	3,695	739
38½	39½							
39½	40½			488			488	98
40½	41½							
		$93,995	$40,817	$123,110	$149,589	$156,394	$563,905	$112,781

of ½ and 1½ years, 1½ and 2½, 2½ and 3½, etc., to the end of property life. Table 15 illustrates such retirement data of water stations taken from the operations in a large railroad system. Columns (1) and (2) show the age intervals; columns (3) to (7) inclusive show the annual retirements during the various age intervals for each year of the five-year period; column (8) shows the total retirements during the five years, and column (9) gives the average retirements per year for the five-year period. It will be noted that the retirements are given in dollars instead of number of units. This is preferred in this case due to the dissimilarity between the property units.

2. Determine the number of units of property of each age in service at the beginning of each year of the period. Then determine the average number for each year of the period. This information will give the average number of units of property in service and ½ year old, the average number of units of property in service and 1½ years old, 2½ years old, etc., to the table end. Sample data showing the value of property in service at each age during the observation period are given in Table 16. The property units are water stations and the period and railroad system are the same as for Table 15. The amount of property in service is given in dollars instead of units for the reason given in connection with the table showing retirements. In the table, column (2) gives the various ages; columns (3) to (7) inclusive give the amounts of property in service at each age and each year from 1922 to 1926 inclusive; column (8) gives the total of columns (3) to (7) inclusive; and column (9) gives the average value of the figures in columns (3) to (7) inclusive. The values in column (9) are obtained by dividing the figures in column (8) by 5.

3. From the data collected under steps 1 and 2 above, the annual retirement rates can be computed as follows: Divide the number of units of property retired between the ages of ½ and 1½ years (called a) by the number of units of property in service and ½ year old (called A). This ratio of a/A will

Table 16. Cost of Water Stations in Service at Each Age for Each Year of the Observation Period

Year Placed	Aver. Age	Property in Service Each Year					Total for Five-Year Period 1922–1926 Incl.	Aver. Cost Five-Year Period 1922–1926 Incl.
		1922	1923	1924	1925	1926		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	Years							
1914	½	$ 97,382	$ 109,879	$ 167,720	$ 178,549	$ 92,532	$ 646,062	$ 129,210
1913	1½	482,054	97,282	109,710	167,720	178,549	1,035,315	207,063
1912	2½	361,632	482,054	96,075	106,989	166,267	1,213,017	242,604
1911	3½	372,693	360,231	480,987	94,625	106,388	1,414,924	282,985
1910	4½	193,924	332,250	360,223	478,774	92,818	1,457,989	291,598
1909	5½	43,906	193,617	328,641	359,929	477,493	1,403,586	280,717
1908	6½	344,386	42,690	193,263	321,886	357,690	1,259,915	251,983
1907	7½	406,709	344,386	42,516	192,728	297,465	1,283,804	256,761
1906	8½	95,360	406,622	343,784	41,927	188,838	1,076,531	215,306
1905	9½	567,253	93,666	403,472	342,258	41,911	1,448,560	289,712
1904	10½	45,407	567,228	92,704	403,472	341,247	1,450,058	290,012
1903	11½	229,259	44,240	565,852	90,280	383,378	1,313,009	262,602
1902	12½	84,747	227,041	44,095	560,248	86,617	1,002,748	200,550
1901	13½	91,145	84,430	225,116	44,095	545,400	990,186	198,037
1900	14½	249,157	90,753	77,109	222,678	41,802	681,499	136,300
1899	15½	407,510	247,439	90,514	75,344	220,653	1,041,460	208,292
1898	16½	204,050	405,754	242,471	90,514	75,270	1,018,050	203,612
1897	17½	69,969	203,500	405,145	240,880	85,741	1,005,235	201,047
1896	18½	131,413	69,553	203,500	404,963	227,581	1,037,010	207,402
1895	19½	92,568	131,413	69,553	199,027	400,958	893,519	178,704
1894	20½	142,238	77,697	129,670	68,165	197,887	615,657	123,132
1893	21½	320,697	139,425	76,652	80,290	66,734	683,798	136,760
1892	22½	51,158	317,742	137,444	72,849	77,464	656,657	131,332
1891	23½	121,575	46,560	313,922	121,290	72,849	676,196	135,239
1890	24½	15,516	118,060	45,558	303,228	102,729	585,091	117,018
1889	25½	45,850	15,516	116,937	45,558	290,499	514,360	102,872
1888	26½	62,377	44,032	15,516	115,800	43,928	281,653	56,331
1887	27½	7,290	62,377	44,032	15,516	114,027	243,242	48,618
1886	28½	44,036	7,290	60,709	44,032	15,516	171,583	34,317
1885	29½	4,827	43,623	7,290	56,790	44,032	156,562	31,312
1884	30½	9,597	4,827	43,065	7,290	53,601	118,380	23,676
1883	31½	37,462	9,597	4,827	43,065	7,290	102,241	20,448
1882	32½	9,635	37,462	9,597	3,240	43,065	102,999	20,600
1881	33½	12,725	7,997	37,462	9,597	3,240	71,021	14,204
1880	34½	5,397	5,678	7,997	37,462	8,081	64,615	12,923
1879	35½	5,767	5,397	3,269	7,997	37,462	59,892	11,978
	36½		5,767	5,397	3,269	7,997	22,430	4,486
	37½	5,487		5,767	5,397	3,269	19,920	3,984
	38½	2,754	5,487		5,767	2,396	16,404	3,281
	39½		972	5,487		5,767	12,226	2,445
	40½			972	5,487		6,459	1,292
	41½	2,006			484	5,487	7,977	1,595
	42½		2,006			484	2,490	498
	43½			2,006			2,006	401
	44½				2,006		2,006	401
	45½					2,006	2,006	401
	46½							
		$5,476,918	$5,491,540	$5,616,026	$5,671,465	$5,614,408	$27,870,357	$5,574,071

give the retirement rate at age ½ year. Likewise divide the number of units retired between the ages of 1½ and 2½ years (b) by the number of units of property in service and 1½ years old (B). This ratio of b/B will give the retirement rate at age 1½ years. These divisions are made for each age to the end of the property life. The annual retirement or mortality rates thus obtained represent the rates at which property will be retired in the succeeding years.

Stated in the language of the insurance actuary the data determined in step 2 give for each age the number of units of property "exposed to the risk of dying," and in step 1 the number of property units of each age which actually did "die." The ratio between the number dying and the number exposed to the risk of dying constitutes the mortality rate for that year of age. Thus a/A represents the rate at which property ½ year old will be retired in the succeeding year, and b/B represents the rate of retirement during the year following an age of 1½ years.

These calculations of annual retirement rates are illustrated in Table 17 for the data on water stations given in Tables 15 and 16. It will be noted that since the retirement rate is a ratio it makes no difference whether the observational data are given in terms of dollars of property cost or numbers of units of property. Columns (1) and (2) give the age interval; column (3) the amount of property retired annually; column (4) the amount of property in existence at the beginning of each age interval; and column (5) the ratio of the amount of property retired to the amount of property in existence.

4. In order to get the mortality table and curve of the class of property assume \$100 worth of new property at age 0. Between ½ and 1½ years of age the amount retired will be the retirement rate a/A times \$100 and the amount of property left at age of 1½ years is \$100 minus $\left(\$100 \times \frac{a}{A}\right)$. Between 1½ and 2½ years the retirement will be b/B times the amount

Table 17. Method of Obtaining Annual Retirement Rates and Percentage of Property Remaining in Service

Age Interval		Average Retirement During Interval	Average Cost of Property in Service at Age in Column (1)	Retirement Rate in Per Cent (3) ÷ (4)	Per Cent Remaining in Service
From	To				
(1)	(2)	(3)	(4)	(5)	(6)
Years	Years				
½	1½	$ 75	$ 129,210	0.058	99.942
1½	2½	1,171	207,063	0.566	99.376
2½	3½	1,552	242,604	0.640	98.740
3½	4½	9,998	282,985	3.533	95.252
4½	5½	1,637	291,598	0.561	94,718
5½	6½	2,404	280,717	0.856	93.907
6½	7½	8,388	251,983	3.329	90.781
7½	8½	1,956	256,761	0.762	90.089
8½	9½	1,338	215,306	0.621	89.530
9½	10½	573	289,712	0.198	89.353
10½	11½	4,890	290,012	1.686	87.847
11½	12½	6,235	262,602	2.374	85.762
12½	13½	3,541	200,550	1.766	84.247
13½	14½	6,049	198,037	3.055	81.673
14½	15½	2,633	136,300	1.932	80.095
15½	16½	2,430	208,292	1.167	79.160
16½	17½	1,549	203,612	0.761	78.558
17½	18½	3,237	201,047	1.610	77.293
18½	19½	1,696	207,402	0.818	76.661
19½	20½	9,504	178,704	5.318	72.584
20½	21½	12,577	123,132	10.214	65.170
21½	22½	2,370	136,760	1.733	64.041
22½	23½	4,914	131,332	3.742	61.645
23½	24½	7,249	135,239	5.360	58.341
24½	25½	3,366	117,018	2.876	56.663
25½	26½	2,500	102,872	2.430	55.286
26½	27½	495	56,331	0.879	54.800
27½	28½	1,376	48,648	2.828	53.250
28½	29½	903	34,317	2.631	51.819
29½	30½	749	31,312	2.392	50.609
30½	31½		23,676		50.609
31½	32½	317	20,448	1.550	49.825
32½	33½	764	20,600	3.709	47.977
33½	34½	1,713	14,204	12.060	42.191
34½	35½	482	12,923	3.730	40.617
35½	36½		11,978		40.617
36½	37½	713	4,486	15,894	34.161
37½	38½	600	3,984	15.060	29.016
38½	39½	739	3,281	22.524	22.480
39½	40½		2,445		22.480
40½	41½	98	1,292	7.585	20.775
41½	42½		1,595		20.775
42½	43½		498		20.775
43½	44½		401		20.775
44½	45½		401		20.775
45½	46½		401		20.775
46½	47½		0		
		$112,781	$5,574,071		

of property left at age of 1½ years. The property left at the age of 2½ years will thus be:

$$100 - \left(100 \times \frac{a}{A}\right) - \left[100 - \left(100 \times \frac{a}{A}\right)\right]\frac{b}{B}.$$

This process is carried on to the limit of the data, thus giving the amount of an initial $100 property value which will be left at each succeeding age. If the observation period

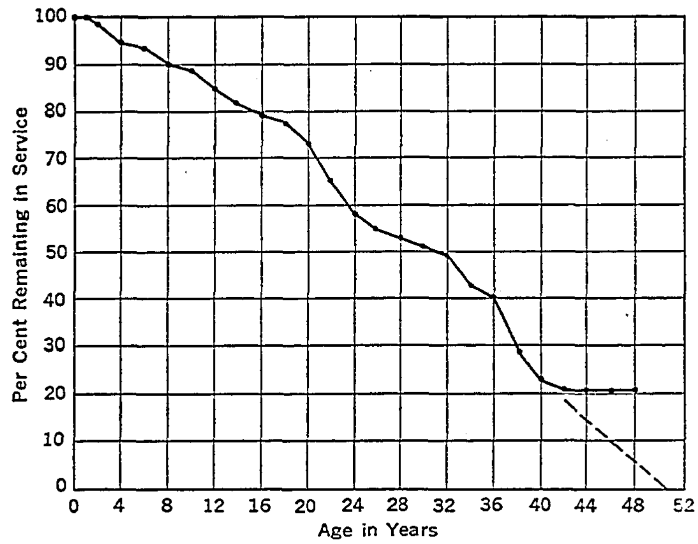


Figure 33. Survivor Curve of Water Stations Obtained by Means of Annual Rate Method Using Five-Year Average

has been properly selected so that retirements have occurred at all ages, there will result a complete life history from observations over only a comparatively short period of time. Instead of assuming $100 worth of property at age 0, it is often preferable to assume 100% as this makes the case more general. The table already referred to also gives in column (6) the percentages of property remaining in service at each age as obtained from the mortality rates given in column (5). It will be noted that the percentage of survivors does not reach zero as it should theoretically. This is due, in this particular case, to the fact that although the railroad had water stations in its system 46½ years old as shown in Table 17, the railroad had no experience as to retirements during the five-year observation period of property older than 41½ years.

Having determined the percentages which survive the various ages, the mortality curve can be drawn. Figure 33 is a plot of the data in column (6) of Table 17. This curve shows the life history for a period of over 40 years based on observations of only five years. The history, however, is not complete since the curve does not drop to the horizontal axis. Property 46½ years old was shown to exist in the company, but the oldest property retired during the five-year period was only 41½ years. Because of this, over 20% of the property is still in service at age 41½. The rates of retirement beyond this age are unknown. If it is assumed that the retirements beyond this age will occur at about the same rate as those immediately preceding this age, the curve can be projected as shown. This extension shows the longest-lived unit to have acquired an age of 50½ years.

Classified List of Mortality Tables of Physical Property.—Following is a list of such mortality tables of physical property as the author has been able to collect from various sources during the past fourteen years. These have been numbered and are listed by classes of property.

I. Water Supply Systems

Number

1–1 Waterworks Sources
2–1 Pumping Stations
3–1 Waterworks Pumps
4–1 Pumping Engines
5–1 Waterworks Boilers

II. Telephone

6–1 Switchboards
7–1 Loading Coils
8–1 Poles
9–1 Central Office Equipment
10–1 Aerial Cable
11–2 Aerial Cable
12–1 Submarine Cable
13–2 Submarine Cable
14–1 Underground Cable
15–2 Underground Cable
16–3 Underground Cable
17–4 Underground Cable
18–5 Underground Cable
19–6 Underground Cable

III. Telegraph

20–1 Poles
21–2 Poles
22–3 Poles
23–4 Poles
24–5 Poles

IV. Electric

25–1 Poles
26–2 Poles
27–1 Gem Lamps
28–2 Incandescent Lamps
29–3 Mazda Lamps
30–4 Mazda B Lamps

V. Railroad

31–1 Car Wheels
32–1 Railway Stations
33–1 Locomotives
34–1 Passenger Cars
35–1 Freight Cars
36–1 Box Cars
37–1 Stock Cars
38–1 Coal Flat Cars
39–1 Beechwood Ties (Tr)
40–2 Beechwood Ties (Tr)
41–3 Cedar Ties (Untr)
42–4 Cedar Ties (Untr)
43–5 Douglas Fir Ties (Tr)
44–6 Douglas Fir Ties (Tr)
45–7 Douglas Fir Ties (Tr)
46–8 Douglas Fir Ties (Tr)
47–9 Hemlock Ties (Tr)
48–10 Hemlock Ties (Tr)
49–11 Hemlock Ties (Tr)
50–12 Hemlock Ties (Tr)
51–13 White Oak Ties (Untr)
52–14 Tamarack Ties (Untr)

It is unnecessary to include more than a few of the 52 tables and their accompanying curves in this book. Figure 34 and Table 18 illustrate the form and method employed in constructing all of the charts and tables.

The original observational data showing the per cent survivors are shown on the chart by the hollow circles. These circles are plotted ½ year ahead of the recorded ages because the interval during which the units are removed actually begins ½ year preceding the recorded ages. A further discussion of this is given in Chapter 6. The per cent survivors shown in the table was obtained by smoothing out the survivor curve and by showing the per cent survivors at the beginning of each age interval instead of at the various ages. This change in the tabulation of the data will be further discussed in Chapter 7.

Table 18. Manner and Form Used in Tabulating Life Characteristics of Electric Poles Shown Graphically in Figure 34

Age Interval	Renewals of Original Units in Age Interval	Survivors at Beginning of Age Interval	Expectancy of Survivors at Beginning of Age Interval	Probable Life of Survivors at Beginning of Age Interval
Years	Per Cent	Per Cent	Years	Years
0– 1	0.1	100.0	11.47	11.47
1– 2	0.4	99.9	10.48	11.48
2– 3	0.9	99.5	9.52	11.52
3– 4	1.6	98.6	8.60	11.60
4– 5	2.0	97.0	7.74	11.74
5– 6	3.8	95.0	6.89	11.89
6– 7	5.7	91.2	6.15	12.15
7– 8	9.5	85.5	5.53	12.53
8– 9	9.5	76.0	5.16	13.16
9–10	10.0	66.5	4.83	13.83
10–11	8.5	56.5	4.59	14.59
11–12	8.0	48.0	4.32	15.32
12–13	7.0	40.0	4.08	16.08
13–14	6.1	33.0	3.84	16.84
14–15	5.0	26.9	3.59	17.59
15–16	4.2	21.9	3.30	18.30
16–17	3.6	17.7	2.97	18.97
17–18	3.0	14.1	2.60	19.60
18–19	2.8	11.1	2.17	20.17
19–20	2.5	8.3	1.73	20.73
20–21	2.3	5.8	1.26	21.26
21–22	2.3	3.5	0.76	21.76
22–22½	1.2	1.2	0.25	22.25
22½–23	0	0	0	22.50

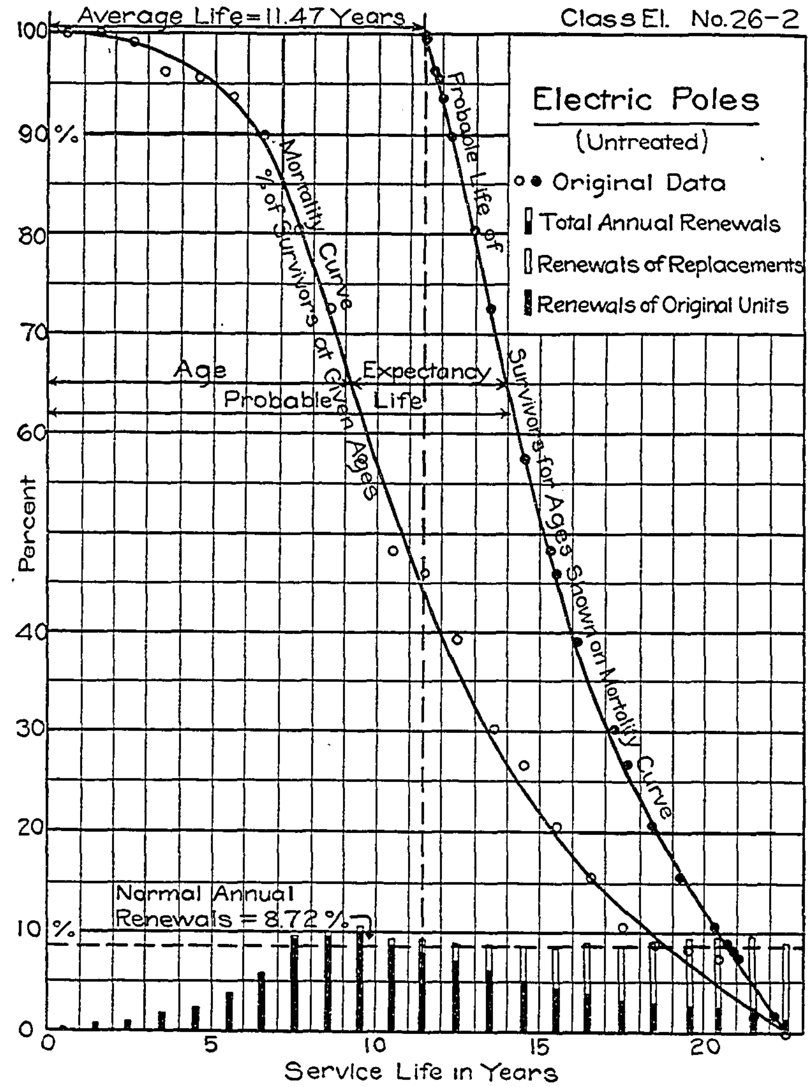


Figure 34. Method of Plotting Life Characteristics of Electric Poles

CHAPTER 3

DEVELOPMENT OF MORTALITY CURVE TYPES

Method of Comparison of Curves.—While compiling and handling the various mortality curves listed in the previous chapter, the similarity in shape of many of the curves became quite noticeable. Some of the curves, however, were noted to be quite steep, while some had a gradual slope, and others were somewhere in between. The grouping of the curves, therefore, suggested itself with the thought that it might be possible to develop a type for each group.

In order that all curves could be easily compared irrespective of the number of years of maximum or average life of the property group, it was necessary to redraw the 52 mortality curves in such a manner that the most important quantity of each group was made equal. This was easily accomplished by making the distance between the 0 age line and the average life line the same for each of the 52 curves. To further simplify the comparison, average life was expressed as 100% life and all other ages were expressed as per cents of average life. By also using per cents to express the number of units surviving, the new 52 mortality curves were identical in all respects except as to the shape and slope of the curves. The chart for the first class of property listed on page 68, redrawn on this basis, is shown in Figure 35.

Selection of Basis Used for Grouping of Curves.—The selection of a satisfactory basis for the development of type curves was difficult. The following bases were tried in the order listed.

A. Curves grouped according to types of property
B. Curves grouped according to slope between 30% and 70% points
C. Curves grouped according to slope between 15% and 85% points
D. Curves grouped according to slope between 20% and 80% points

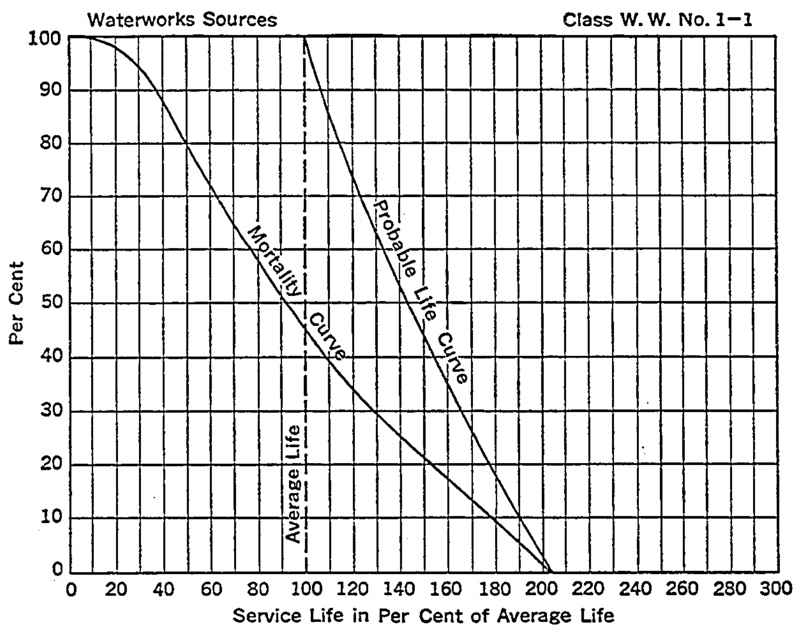


Figure 35. Mortality and Probable Life Curves with Average Life Equal to 100%, and Other Ages Expressed as Percentages Thereof

E. Curves grouped according to maximum life
F. Curves grouped according to inspection
G. Curves grouped according to slope between 25% and 75% points

The grouping by classes of property gave no uniform gradation of types and was immediately dropped. The results of this comparison of curves of similar classes of property are of interest, however, and the groupings are reproduced in the latter part of this chapter. Since the shape or appearance of a mortality curve appears to be largely determined by its slope, a study was made using the slopes between two points on the curve as the basis of classification. Three studies were made on this basis using the slopes between the 70% and 30%, 85% and 15%, and 80% and 20% points. This method of grouping showed promise and was ultimately adopted. Grouping by per cent maximum life as a basis was also investigated but with little success. Sorting by sight or inspection was not satisfactory in itself but was used in combination with the slope method already mentioned. The slope finally used was that existing between the 75% and 25% points on the curve. The slope was expressed in terms of the per cent of average life between the two points.

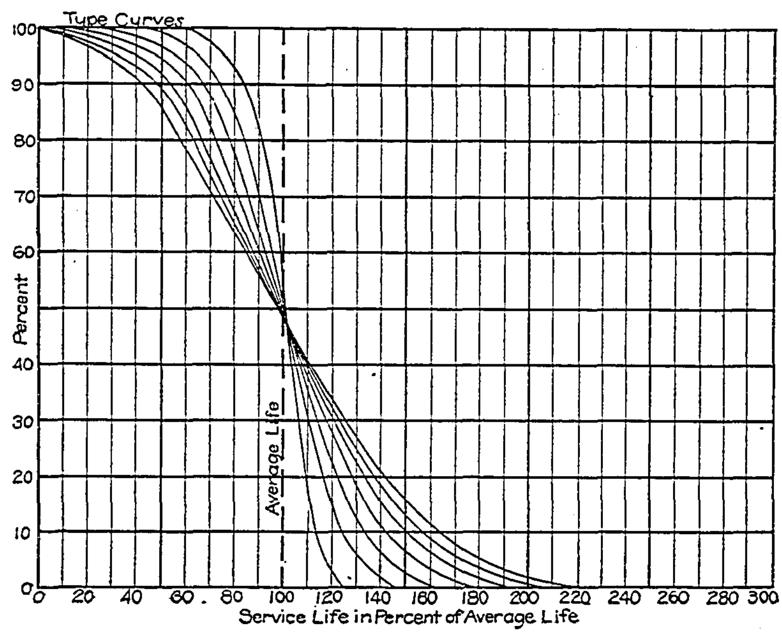


Figure 36. Seven Type Curves Obtained by Grouping 52 Mortality Curves According to Slope Between 75% and 25% Points

Developing Types.—Classification of the 52 curves resulted in the formation of seven distinct groups. Some groups only contained a few curves and others contained as many as 17. The type curve for each group was obtained by averaging the curves in the group. The average curves thus obtained were then drawn on the same sheet and changes made so that the gradation from one type to the other was reasonably uniform. Following this it was necessary further to readjust the curves in order that their included areas would be the same and of such value as to give an average life equal to 100%. When this adjustment was secured the seven type curves shown in Figure 36 resulted. The survivor data for these curves are given in Table 19.

Table 19. Per Cent Surviving Various Ages for Each of Seven Type Mortality Curves

Age Interval	Per Cent Surviving at Beginning of Age Intervals						
	I	II	III	IV	V	VI	VII
0– 10	100.00	100.00	100.00	100.00	100.00	100.00	100.00
10– 20	100.00	100.00	100.00	100.00	99.50	98.60	98.40
20– 30	100.00	100.00	100.00	99.50	98.50	97.40	96.60
30– 40	100.00	100.00	99.70	98.30	97.10	95.50	94.20
40– 50	100.00	100.00	98.80	96.90	95.20	93.20	91.10
50– 60	100.00	99.40	97.00	94.50	92.20	89.70	85.50
60– 70	100.00	97.30	94.00	90.60	86.10	82.70	78.60
70– 80	97.60	93.20	88.30	82.90	77.60	74.00	70.60
80– 90	93.10	85.40	78.00	71.40	67.60	65.00	63.00
90–100	82.00	70.10	63.90	59.90	57.80	56.20	55.50
100–110	55.00	51.00	50.00	49.20	49.00	48.50	48.30
110–120	19.50	31.50	36.20	38.50	39.60	40.00	41.00
120–130	3.00	15.00	23.00	28.20	31.00	32.40	34.00
130–140	0.00	6.00	12.60	18.50	22.50	24.80	26.40
140–150		1.40	6.30	11.10	15.70	18.10	20.80
150–160		0.00	2.60	6.40	10.10	13.10	15.90
160–170			0.40	3.60	6.10	9.20	11.50
170–180			0.00	1.70	3.70	6.00	7.90
180–190				0.00	1.30	3.50	5.50
190–200					0.00	1.80	3.10
200–210						0.50	1.80
210–220						0.00	0.50
220–230							0.00

Significance of Type Curves.—The significance of these seven type curves is that they cover the range of typical mortality curves of physical property found to date. Only abnormal or otherwise eccentric mortality curves lie beyond the range encompassed by these seven type curves. Moreover, these seven type curves are representative of practically all mortality curves of physical property. Any laws or relations found to hold for these type curves, therefore, may be assumed to hold for the entire field of physical property. The latter is indeed the most important aspect of these type curves, and much of the discussion in the following chapters centers around the further development of these representative curves.

Relation of Types to Classes of Property.—The type mortality curve which each of the 52 property groups best fits is shown in Table 20. It should be mentioned that it was somewhat difficult to classify several of the curves as their shape differed too widely from any of the types to make classification reliable. The classification is therefore not final nor conclusive; in fact, it is easily possible that mortality curves of the same class of property may not be very identical due to different operating conditions, managerial policies, climate, state of the art, etc.

It is of interest, however, to note in a general way the types into which the different classes of property fall; thus, waterworks property seems to fall into the middle and upper types, III and VII. The same is true of telephone poles, cables, central office equipment, electric poles and lamps. These fall largely into Types VI and VII with a few in V. Railroad equipment, however, such as locomotives, passenger cars, freight cars, flat cars, stock cars, and ties seem to fall into the lower type classes, principally II and III. Ties make up the bulk of the curves falling in Type II with a few in Types I and III. The latter groups of property mentioned are more like the "one-hoss shay" in that they almost all live the same

Table 20. Relation of Classes of Property to Type Mortality Curves

(Table shows type curve which best fits each property group.)

Number	Description	Type						
		I	II	III	IV	V	VI	VII
1–1	Waterworks sources							X
2–1	Pumping stations							X
3–1	Waterworks pumps							X
4–1	Pumping engines			X				
5–1	Waterworks boilers				X			
6–1	Switchboards							X
7–1	Loading coils							X
8–1	Poles							X
9–1	Central office equipment							X
10–1	Aerial cable							X
11–2	Aerial cable						X	
12–1	Submarine cable							X
13–2	Submarine cable							X
14–1	Underground cable						X	
15–2	Underground cable						X	
16–3	Underground cable			X				
17–4	Underground cable						X	
18–5	Underground cable							X
19–6	Underground cable							X
20–1	Poles						X	
21–2	Poles						X	
22–3	Poles							X
23–4	Poles					X		
24–5	Poles					X		
25–1	Poles							X
26–2	Poles					X		
27–1	Gem lamps					X		
28–2	Incandescent lamps							X
29–3	Mazda lamps		X					
30–4	Mazda B lamps				X			
31–1	Car wheels							X
32–1	Railway stations							X
33–1	Locomotives				X			
34–1	Passenger cars			X				
35–1	Freight cars			X				
36–1	Box cars		X					
37–1	Stock cars			X				
38–1	Coal flat cars			X				
39–1	Beechwood ties		X					
40–2	Beechwood ties			X				
41–3	Cedar ties	X						
42–4	Cedar ties	X						
43–5	Douglas fir ties		X					
44–6	Douglas fir ties		X					
45–7	Douglas fir ties		X					
46–8	Douglas fir ties		X					
47–9	Hemlock ties		X					
48–10	Hemlock ties		X					
49–11	Hemlock ties		X					
50–12	Hemlock ties		X					
51–13	White oak ties			X				
52–14	Tamarack ties		X					
	Totals	2	12	8	3	4	6	17

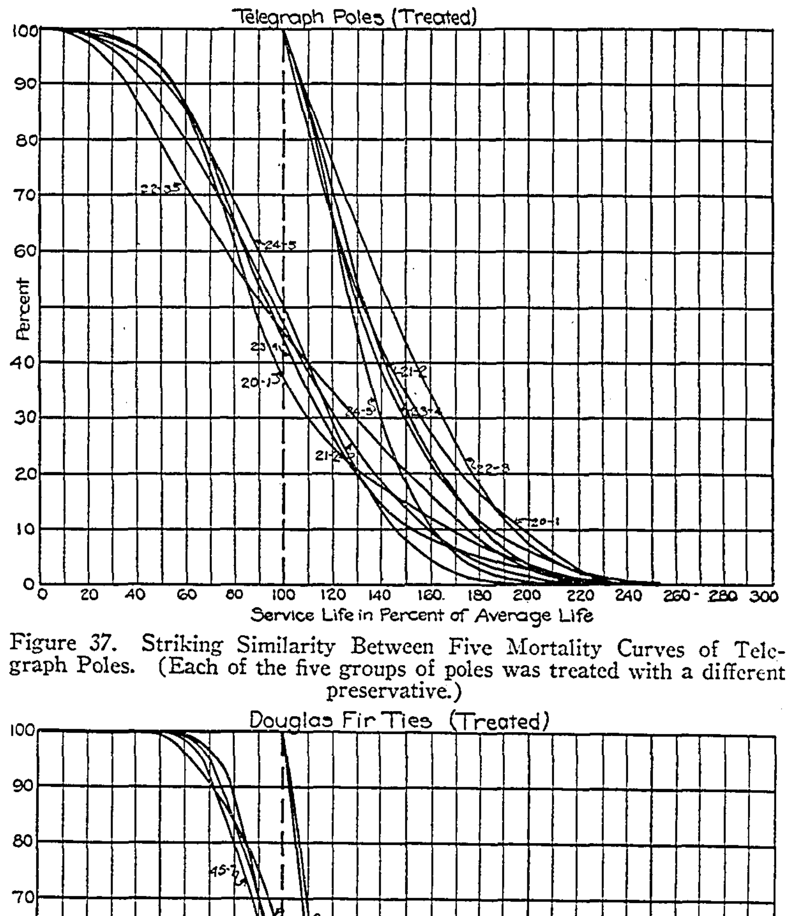


Figure 37. Striking Similarity Between Five Mortality Curves of Telegraph Poles. (Each of the five groups of poles was treated with a different preservative.)

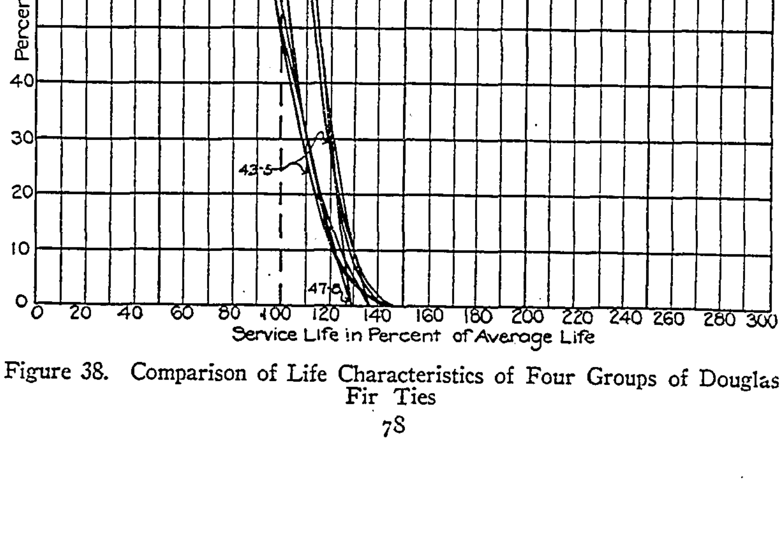


Figure 38. Comparison of Life Characteristics of Four Groups of Douglas Fir Ties

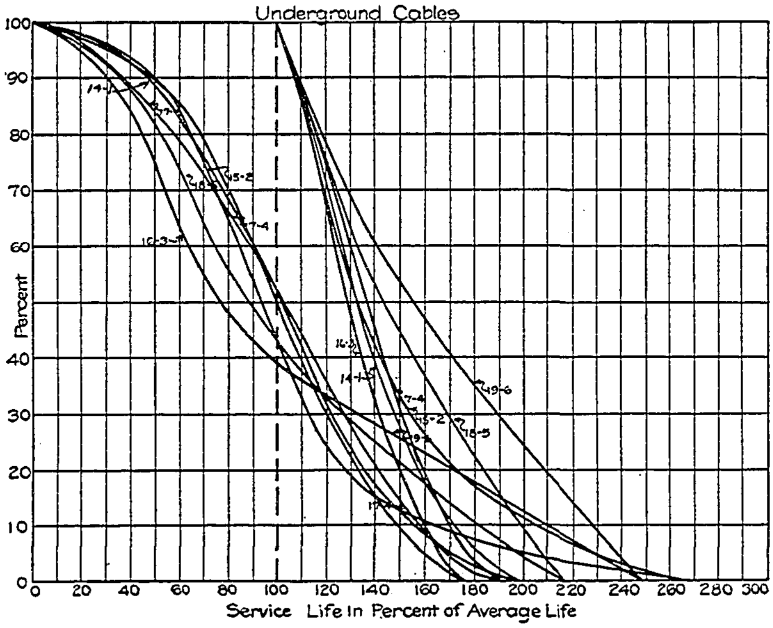


Figure 39. Comparison of Life Characteristics of Six Groups of Underground Cables

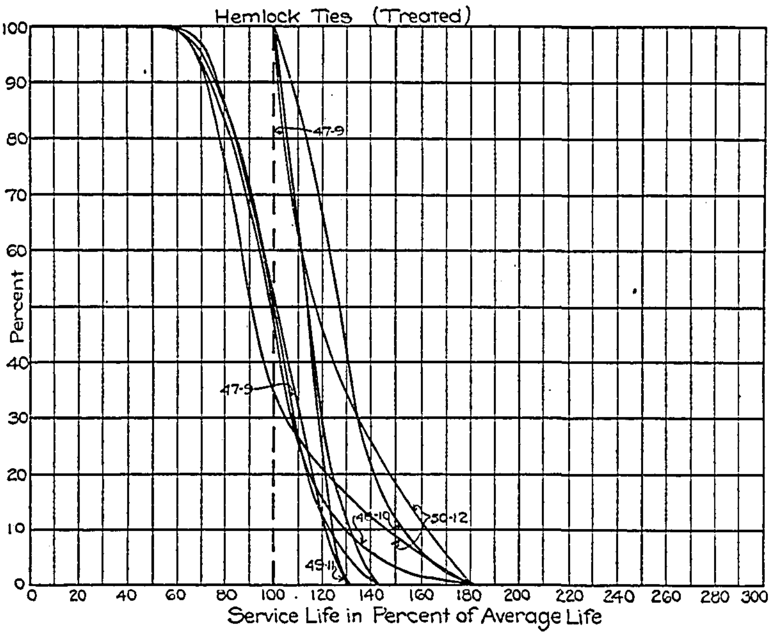


Figure 40. Comparison of Life Characteristics of Four Groups of Hemlock Ties

life and then go out of service simultaneously, whereas the former groups are no doubt more subject to inadequacy and obsolescence and are therefore often removed in the early years of service.

It will be noted in examining Table 20 that the property classes as shown run approximately from right to left in types.

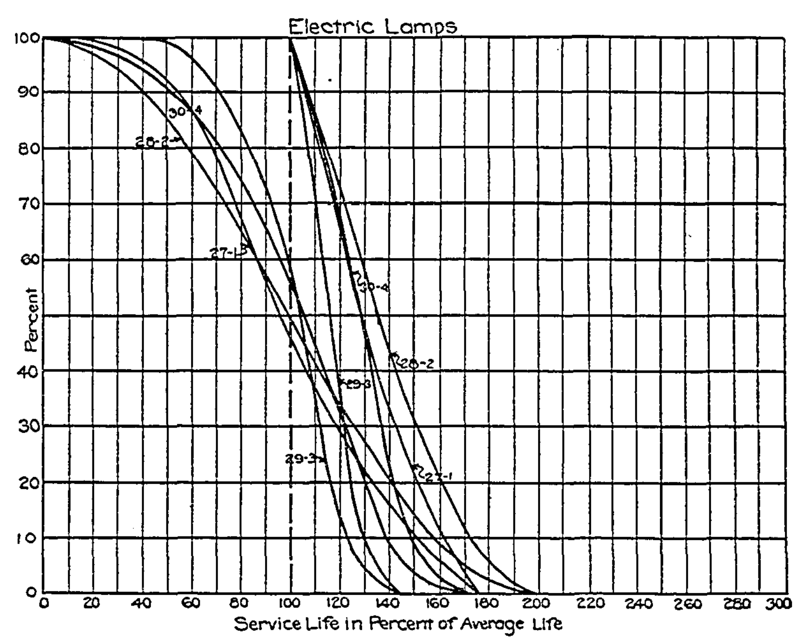


Figure 41. Comparison of Life Characteristics of Four Groups of Electric Lamps

Waterworks property is at the extreme right, next is electric property, then comes railroad rolling stock, and finally ties at the extreme left.

Grouping by Classes of Property.—The comparisons resulting from the grouping of similar property are given in Figures 37 to 41 inclusive. Figure 37 shows five mortality curves of telegraph poles drawn so as to have the same average life on a percentage scale. The curves show striking marks of similarity even though each group of poles was treated with a different preservative. In like manner the curves showing Douglas fir ties, underground cables, hemlock ties, and electric lamps, in each case show remarkable similarity. In a few instances one of the curves on a chart is different from the others. This variation, however, no doubt could be accounted for if all the pertinent facts were known.

CHAPTER 4

FREQUENCY CURVES OF REPLACEMENT OF ORIGINAL UNITS

Renewals of Original Units.—The simplest method of compiling mortality statistics, as discussed in Chapter 2, consists in recording the age of each unit as it goes out of service, and then after the lives of a sufficient number of units of the same class have thus been recorded to summarize the data in a table such as Table 21.

Each mark in the second column represents one unit. When all of the units have been entered in the table, the number that have reached each age are added. If columns 1 and 3 are then plotted as shown in Figure 42, the common distribution or frequency curve results. The curve shows how many units of a given group are removed after reaching various ages in service.

Such a distribution curve has been prepared for each of the groups of physical property listed in Chapter 2. These curves are shown near the lower edge of the diagram chart, as shown in Figure 43. The portions appearing in solid white are the percentages of original units replaced each year. The bars are drawn at the middle of each age interval, thus, 0–1, 1–2, 2–3, etc., and show the percentage of original units removed during the age interval.

Types of Distribution Curves.—It was noted by examining the distribution curves for the 52 groups of property that the curves are somewhat similar and have the following characteristics in common. Each curve starts at or near zero, rises to a maximum, and then drops back to zero. Some reach the maximum or peak in the early life of the group and others in

Table 21. Data on Removal from Service

Age When Removed from Service	Number of Units Removed from Service after Reaching Ages Shown in Opposite Column	
1.	1	= 1
2.	111	= 3
3.	~~1111~~ 1	= 6
4.	~~1111~~ ~~1111~~	= 10
5.	~~1111~~ 11	= 7
6.	1111	= 4
7.	11	= 2
8.	1	= 1
		34

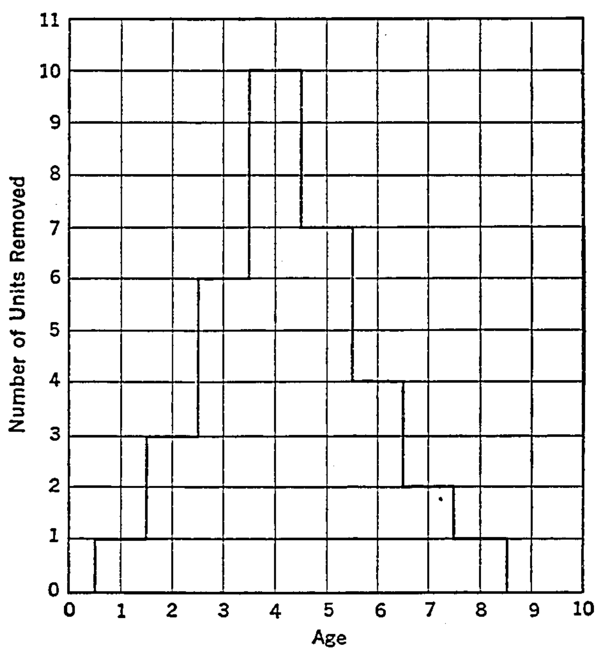


Figure 42. Typical Distribution Curve

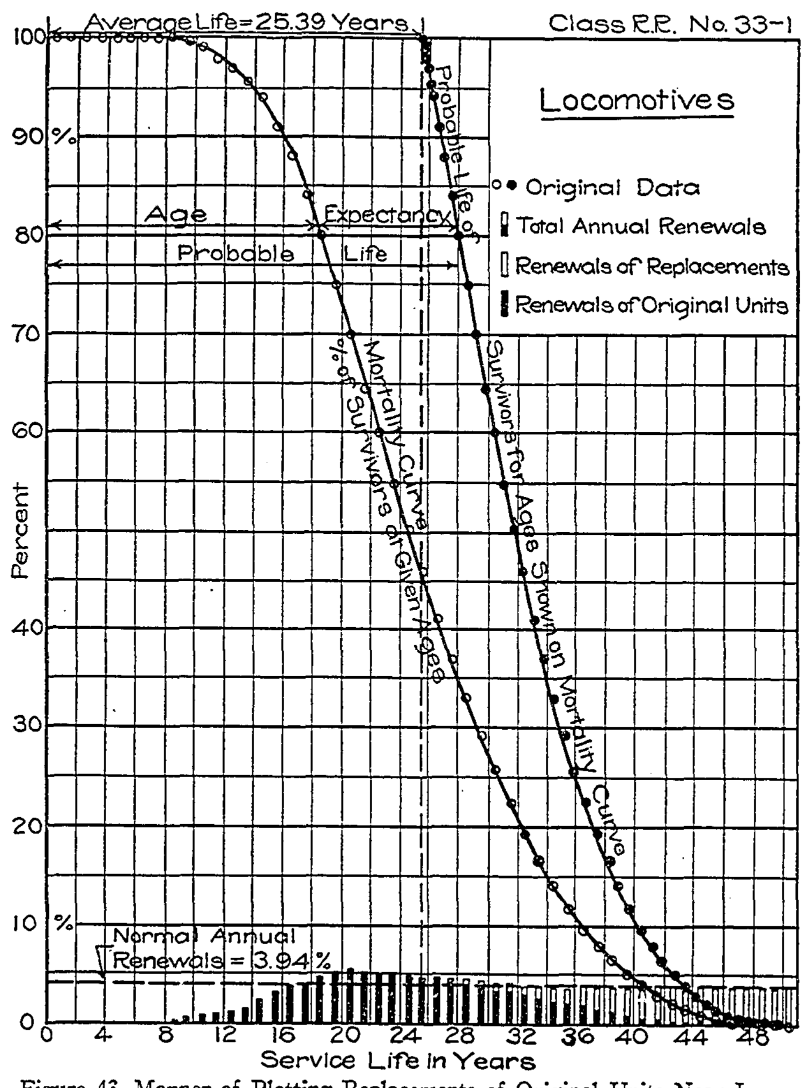


Figure 43. Manner of Plotting Replacements of Original Units Near Lower Edge of Chart. (Black portions of vertical bars represent number of units of original group replaced each year.)

the later life of the group. Some have a large percentage of replacement in the earlier years of the life of the group, and others do not have any.

It is apparent that the shapes of the distribution curves are determined by the shapes of the mortality curves and vice versa. If the mortality curve drops off rapidly the replacements in the earlier years are large. However, if the mortality curve is horizontal during the early life of the group the replacements during that period are small. Likewise, if the mortality curve is steep, during the latter years the renewals during that period are large; and if its slope is gradual then the renewals are more nearly normal.

The determination of distribution or frequency curve types for the 52 groups of physical property is, therefore, dependent upon the types of mortality curves. The types of mortality curves have, however, already been developed in Chapter 3, and the types of distribution curves corresponding to these type mortality curves can be obtained directly from the percentage survivor data.

Table 22 gives the percentage survivor data in columns (1), (3), (5), (7), (9), (11), and (13) for the seven type mortality curves developed in Chapter 3, and the corresponding replacement data in columns (2), (4), (6), (8), (10), (12), and (14). The per cent survivors are recorded at the beginning of each age interval, and the per cent replacements are those made during the age interval.

These percentage replacements for the seven type frequency curves are plotted in Figure 44. This chart shows a decrease in the peak values of the replacements in Types I to VII from 35.50% for Type I to 8% for Type VII. This decrease in peak value is accompanied by a spreading out of the replacements over a longer period of time. In Type I all of the replacements are made over a period of 70% of average life, while in Type VII the replacements are made over a period

Table 22. Survivor and Replacement Data for Seven Type Curves Developed in Chapter 3

Survivor data in columns (1), (3), (5), (7), (9), (11), and (13). Replacement data in columns (2), (4), (6), (8), (10), (12), and (14).

Age Interval	Types													
	I		II		III		IV		V		VI		VII	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
0- 10	100.00	0.00	100.00	0.00	100.00	0.00	100.00	0.00	100.00	0.50	100.00	1.40	100.00	1.60
10- 20	100.00	0.00	100.00	0.00	100.00	0.00	100.00	0.50	99.50	1.00	98.60	1.20	98.40	1.80
20- 30	100.00	0.00	100.00	0.00	100.00	0.30	99.50	1.20	98.50	1.40	97.40	1.90	96.60	2.40
30- 40	100.00	0.00	100.00	0.00	99.70	0.90	98.30	1.40	97.10	1.90	95.50	2.30	94.20	3.10
40- 50	100.00	0.00	100.00	0.60	98.80	1.80	96.90	2.40	95.20	3.00	93.20	3.50	91.10	5.60
50- 60	100.00	0.00	99.40	2.10	97.00	3.00	94.50	3.90	92.20	6.10	89.70	7.00	85.50	6.90
60- 70	100.00	2.40	97.30	4.10	94.00	5.70	90.60	7.70	86.10	8.50	82.70	8.70	86.60	8.00
70- 80	97.60	4.50	93.20	7.80	88.30	10.30	82.90	11.50	77.60	10.00	74.00	9.00	70.60	7.60
80- 90	93.10	11.10	85.40	15.30	78.00	14.10	71.40	11.50	67.60	9.80	65.00	8.80	63.00	7.50
90-100	82.00	27.00	70.10	19.10	63.90	13.90	59.90	10.70	57.80	8.80	56.20	7.70	55.50	7.20
100-110	55.00	35.50	51.00	19.50	50.00	13.80	49.20	10.70	49.00	9.40	48.50	8.50	48.30	7.30
110-120	19.50	16.50	31.50	16.50	36.20	13.20	38.50	10.30	39.60	8.60	40.00	7.60	41.00	7.00
120-130	3.00	3.00	15.00	9.00	23.00	10.40	28.20	9.70	31.00	8.50	32.40	7.60	34.00	7.60
130-140	0.00		6.00	4.60	12.60	6.30	18.50	7.40	22.50	6.80	24.80	6.70	26.40	5.60
140-150			1.40	1.40	6.30	3.70	11.10	4.70	15.70	5.60	18.10	5.00	20.80	4.90
150-160			0.00		2.60	2.20	6.40	2.80	10.10	4.00	13.10	3.90	15.90	4.40
160-170					0.40	0.40	3.60	1.90	6.10	2.40	9.20	3.20	11.50	3.60
170-180					0.00		1.70	1.70	3.70	2.40	6.00	2.50	7.90	2.40
180-190							0.00		1.30	1.30	3.50	1.70	5.50	2.40
190-200									0.00		1.80	1.30	3.10	1.30
200-210											0.50	0.50	1.80	1.30
210-220											0.00		0.50	0.50
220-230													0.00	

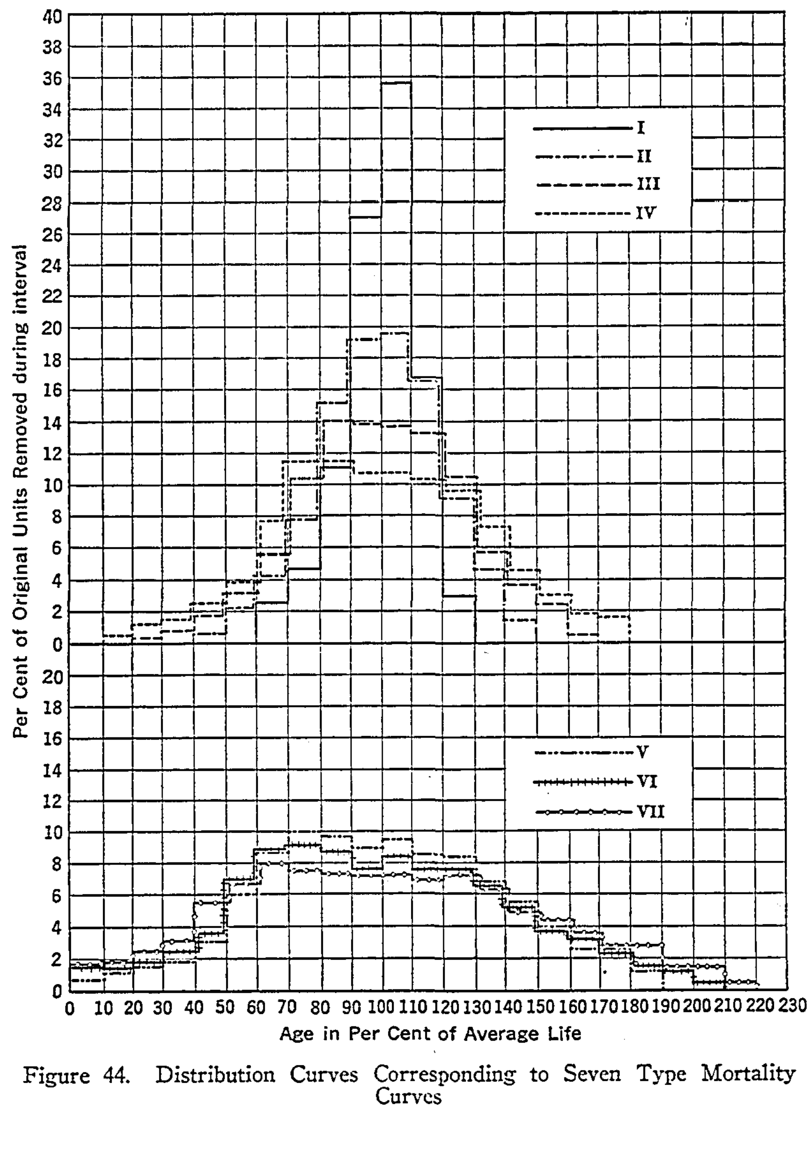


Figure 44. Distribution Curves Corresponding to Seven Type Mortality Curves

of 220% of average life. A swing in the peak value from later life is also noticeable in going from Types I to VII.

General Equation of Frequency Curves.—The various forms which a frequency curve may assume are twelve in number.[1] Of these the following seven are the most common. Types I, IV, and VI are known as the main types and the others are known as the transition types.

I. Limited range in both directions, and skew.
II. Limited range in both directions, and symmetrical.
III. Limited range in one direction and skew.
IV. Unlimited range and skew.
V. Limited range in one direction and skew.
VI. Limited range in one direction and skew.
VII. Unlimited range and symmetrical.

The general equation which will fit any of these forms of a frequency curve is:

$$\frac{dy}{dx} = \frac{y(x + a)}{f(x)}$$

in which $f(x)$ is any function of x. The conditions which this general form satisfies are:

1. The frequency curve must have a maximum point; that is, for some finite value of x, $\frac{dy}{dx}$ must equal zero. This condition is satisfied, for when $x = -a$,

$$\frac{dy}{dx} = \frac{y(-a + a)}{f(x)} = 0$$

This shows that the tangent to the curve at $x = -a$ is horizontal and this is only possible at a maximum or minimum point. The distance $(-a)$ is, therefore, the distance from the origin to the mode or peak value.

[1] These frequency curves were developed by Professor Karl Pearson and later in 1906, W. Palin Elderton prepared a manuscript for the *Journal of the Institute of Actuaries* on "Frequency Curves and Correlation." In this treatise Mr. Elderton gives a thorough discussion of the basis and practical application of the Pearsonian frequency curves to actuarial data.

2. The frequency curve is likely to be tangent to the X-axis at one end, and possibly both ends. This condition is also satisfied by the equation, for when $y = 0$,

$$\frac{dy}{dx} = \frac{O(x + a)}{f(x)} = 0 \quad \text{(Provided } f(x) \text{ does not vanish)}$$

again showing that the tangent to the curve is horizontal, or what amounts to the same thing, the horizontal or x axis is tangent to the curve.

By means of Maclaurin's theorem $f(x)$ can be expanded, which will put the equation into the form:

$$\frac{1}{y}\frac{dy}{dx} = \frac{x + a}{b_0 + b_1x + b_2x^2 + \text{etc.}}$$

Although the expression for $f(x)$ is a series of indefinite length, it is usually not necessary to extend the series beyond the term b_2x^2.

The Criterion K.—It will be noted that the general equation is a differential equation. To obtain the equation of the curve, free from differentials, therefore, the expression must be integrated. The integration will depend on the nature of the roots of the expression $b_0 + b_1x + b_2x^2 + \text{etc.} = 0$. By factoring this expression it will be seen that the nature of the roots depends on the value of the expression $\frac{b_1^2}{4b_0b_2}$. Karl Pearson showed how these constants can be computed from the values of the average of the frequency series, and the squares, cubes, and fourth powers of the deviations from the average. He calls these last four quantities μ_1, μ_2, μ_3, and μ_4, respectively. Then he lets

$$B_1 = \frac{\mu_3^2}{\mu_2^3} \quad \text{and} \quad B_2 = \frac{\mu_4}{\mu_2^2}$$

and finally $$\frac{b_1^2}{4b_0b_1} = \frac{B(B + 3)}{4(4B_2 - 3B_1)\,(2B_2 - 3B_1 - 6)} = K$$

Table 23. Equations of Pearsonian Frequency Curves and Their Respective Criteria

No. of type usually adopted (Pearson).	Equation to Curve in form usually adopted (Pearson). Equation.	Origin.	Criterion.	Remarks.
MAIN TYPES.				
I.	$y=y_0(1+x/a_1)^{\nu a_1}(1-x/a_2)^{\nu a_2}$	mode	κ negative	Limited range; skew; usually bell-shaped, but may be U-shaped, J-shaped or twisted J-shaped
IV.	$y=y_0(1+x^2/a^2)^{-m}e^{-\nu\tan^{-1}x/a}$	$\nu a/(2m-2)$ after mean	$\kappa>0$ and <1	Unlimited range; skew; bell-shaped
VI.	$y=y_0(x-a)^{q_2}x^{-q_1}$	a before start of curve	$\kappa>1$	Unlimited range in one direction; skew; bell-shaped, but may be J-shaped.
TRANSITION TYPES.				
"Normal Curve."	$y=y_0e^{-x^2/2\sigma^2}$	mode (=mean)	$\kappa=0,\ \beta_1=0,\ \beta_2=3$	Unlimited range; symmetrical; bell-shaped
II.	$y=y_0(1-x^2/a^2)^m$	mode (=mean)	$\kappa=0,\ \beta_1=0,\ \beta_2<3$	Limited range; symmetrical; usually bell-shaped, but U-shaped when $\beta_2<1{\cdot}8$
VII.	$y=y_0(1+x^2/a^2)^{-m}$	mode (=mean)	$\kappa=0,\ \beta_1=0,\ \beta_2>3$	Unlimited range; symmetrical; bell-shaped
III.	$y=y_0(1+x/a)^{\gamma a}e^{-\gamma x}$	mode	$2\beta_2=6+3\beta_1$	Unlimited range in one direction; usually bell-shaped, but may be J-shaped
V.	$y=y_0x^{-p}e^{-\gamma/x}$	start of curve	$\kappa=1$	Unlimited range in one direction; bell-shaped
VIII.	$y=y_0(1+x/a)^{-m}$	end of curve	κ negative; $\lambda=0$; $5\beta_2-6\beta_1-9$ negative	Range from infinite ordinate at $-a$ to finite ordinate at 0 (or from $a(1-m)/(2-m)$ to $a/(2-m)$ with origin at mean)
IX.	$y=y_0(1+x/a)^m$	end of curve	κ negative; $\lambda=0$; $5\beta_2-6\beta_1-9$ positive, $2\beta_2-3\beta_1-6$ negative	Range from $x=-a$ where $y=0$ to $x=0$ where $y=y_0$ (or from $-a(m+1)/(m+2)$ to $a/(m+2)$, with origin at mean)
X.	$y=y_0e^{-x/\sigma}$	start of curve	$\beta_1=4,\ \beta_2=9$	Exponential from finite ordinate at 0 (or $-\sigma$ with origin at mean) to infinitesimal ordinate at ∞; J-shaped
XI.	$y=y_0x^{-m}$	b before start	$\kappa>1,\ \lambda=0,\ 2\beta_2-3\beta_1-6$ positive	J-shaped; starts at $x=b$ (or $-b/(m-2)$ with origin at mean) where ordinate is finite
XII.	$y=y_0\left(\frac{\sigma(\sqrt{3+\beta_1}+\sqrt{\beta_1})+x}{\sigma(\sqrt{3+\beta_1}-\sqrt{\beta_1})-x}\right)^{\sqrt{\beta_1/3+\beta_1}}$	mean	$5\beta_2-6\beta_1-9=0$	Twisted J-shaped; special case of Type I

The values of B_1, B_2, and K determine to which of the forms enumerated in the preceding article the given frequency curve belongs, and the exact form of the equation which will probably fit that form. Table 23 shows the twelve classes, the form of equation for each class, and the values of B_1, B_2, and K which fix the class into which a given frequency curve falls. The values of K and the corresponding types are also shown in Figure 45. This figure makes the classification of frequency curves a simple process. A quotation from Mr. Elderton's book indicates the significance of this criterion:

"If this expression for the criterion is negative the roots are real and of different sign and we get one of the main types

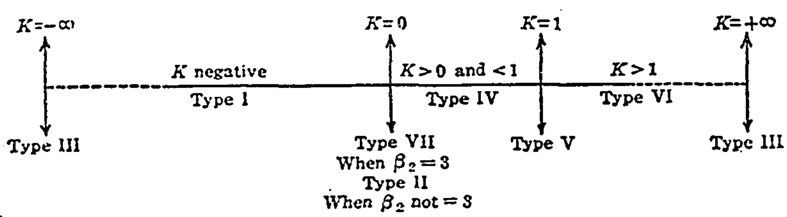


Figure 45. Values of Criterion K for Various Type Frequency Curves

of curve—called Type I by Professor Karl Pearson, to whom this system of curves is due—if the expression is positive and less than unity the roots are complex, and we get the second main type (Pearson's Type IV) and if the expression is positive and greater than unity the roots are real and of the same sign, and we reach the third main type (Pearson's Type VI).

"This really covers the whole field, but in the limiting cases when one type changes into another we reach simpler forms of transition curves. Thus when the criterion is large (theoretically infinite) one root is infinite (Type III), when it is unity the two roots are equal (Type V) and when it is zero the roots are equal in magnitude but of opposite sign (Type II). If in the last case $b_1 = b_2 = 0$, we reach what we shall call the 'normal curve of error.'"

Table 24. Moment Calculations for No. III Frequency Group

Age Interval	Replacements	Distance from Mean	First Moments −	First Moments +	Second Moments +	Third Moments −	Third Moments +	Fourth Moments +
0– 10	0.00							
10– 20	0.00							
20– 30	0.30	− 7½	2.25		16.875	126.56		949.21
30– 40	0.90	− 6½	5.85		38.025	247.17		1,606.55
40– 50	1.80	− 5½	9.95		54.450	299.48		1,647.11
50– 60	3.00	− 4½	13.50		60.750	273.36		1,230.18
60– 70	5.80	− 3½	20.30		71.050	248.70		870.36
70– 80	10.30	− 2½	25.75		64.375	160.89		402.34
80– 90	14.10	− 1½	21.15		31.725	47.66		71.38
90–100	13.90	− ½	6.95		3.475	1.74		0.87
100–110	13.80	½		6.90	3.450		1.73	0.86
110–120	13.20	1½		19.80	29.700		44.55	66.82
120–130	10.40	2½		26.00	65.000		162.50	406.25
130–140	6.20	3½		21.70	75.950		265.82	930.38
140–150	3.70	4½		16.65	74.925		337.16	1,517.23
150–160	2.20	5½		12.10	66.550		366.02	2,013.14
160–170	0.40	6¼		2.50	15.624		97.66	610.28
			105.70	105.65	671.924	1,405.56	1,275.44	12,322.96
						1,275.44		
						130.12		

Calculation of Criteria for Type Frequency Curves of Physical Property.—In order to determine to which type of theoretical frequency curve the actual physical property frequency curves belong it is necessary first to evaluate the first, second, third, and fourth moments as the values of μ_1, μ_2, μ_3, and μ_4 fix the values of B_1 and B_2, and B_1 and B_2 in turn fix the value of K. The center of moments is the mean. The first moment is thus obtained by multiplying the replacements in each interval by the distance between the given interval and the mean. The second moment is obtained by multiplying the replacements by the square of the distance or arm, the third moment by the cube of the distance from the mean, and the fourth moment by the fourth power of the distance. If then the total first moment products are divided by the total replacements, the first moment μ_1 results. Likewise, if the total second moment products are divided by the total replacements the second moment μ_2 results. The third and fourth moments are computed in similar manner. Table 24 illustrates these moment calculations for No. III frequency group.

It will be noted that since the moments are taken about the mean or gravity axis of the area, the first moment $\mu_1 = 0$. This is shown by the equality in positive and negative moment products. In a few instances it was necessary to transfer a few tenths of replacements from one side of the mean to the other to bring about this equality. In the illustration, 6.30 was reduced to 6.20, and 5.70 was increased to 5.80. It should also be noted that all the moment products of the second and fourth moments are positive as the square or fourth power of any negative member is positive. The third moment products, however, may be either positive or negative in accordance with the sign of the moment arm. The net moment product is the difference between the total positive and negative moment products. The values for μ_2, μ_3, and μ_4 for this illustration are thus:

$$\mu_2 = \frac{671.92}{100} = 6.7192$$

$$\mu_3 = \frac{130.12}{1,000} = -1.3012$$

$$\mu_4 = \frac{12,322.96}{100} = 123.23$$

Table 25 gives the values for μ_2, μ_3, and μ_4 for the seven frequency groups. It will be noted that the moments increase in magnitude from Types 1 to VII, thus: μ_2 increases from 1.462 to 22.09; μ_3 from —1.1939 to 25.965; and μ_3 from 7.4143 to 1,154.5. Sheppard's adjustments were not applied to any of the moments, as the curves did not appear to have high contact at either end.

Table 25. Moments and Criteria for the Seven Frequency Groups

Type	Moments			Criteria		
	μ_2	μ_3	μ_4	B_1	B_2	K
I.......	1.462	− 1.1939	7.4143	0.45585	3.4673	− 5.2440
II.......	3.7692	− 1.3772	40.014	0.03527	2.8165	− 0.05653
III.......	6.7192	− 1.3012	123.23	0.00558	2.7294	− 0.075313
IV.......	10.287	0.5428	279.92	0.00027	2.645	− 0.0002866
V.......	13.934	2.0904	485.92	0.001615	2.5028	− 0.001223
VI.......	17.111	13.367	737.92	0.03649	2.5202	− 0.026076
VII.......	22.09	25.965	1,154.5	0.0625	2.3647	− 0.03326

The calculations for B_1 and B_2 follow directly from the values of μ_2, μ_3, and μ_4, thus:

$$B_1 = \frac{\mu_3^2}{\mu_2^3} \text{ and } B_2 = \frac{\mu_4}{\mu_2^2}$$

Substituting the values of moments obtained above for frequency group III gives:

$$B_1 = \frac{(1.3012)^2}{(6.7192)^3} = 0.0055813$$

$$B_2 = \frac{123.23}{(6.7192)^2} = 2.7294$$

In a similar manner the values for B_1 and B_2 for the other six groups were calculated and are shown in Table 25.

The value of the criterion K can now be obtained by substituting in the expression for K thus:

$$K = \frac{B_1(B_2+3)^2}{4(4B_2-3B_1)(2B_2-3B_1-6)}$$

$$= \frac{0.0055813(5.7294)^2}{4(10.9176-0.0167439)(5.4588-0.0167439-6)}$$

$$= -0.075313$$

The values of K for the seven frequency groups are also given in Table 25 along with the values of moments and criteria B_1 and B_2.

Since the value of K is negative in each instance, the proper type of equation to use is Type I, as shown by Table 23 and Figure 45. The equation for this type of frequency curve is:

$$y = y_0\left(1+\frac{x}{a_1}\right)^{m_1}\left(1-\frac{x}{a_2}\right)^{m_2}$$

This equation represents a curve which is skew in form and has a limited range in both directions. Its origin is at the mode. The evaluation of the constants in the equation is discussed in the following paragraph.

Evaluation of Frequency Equation Constants.—The constants which must be evaluated for the Type I frequency equation are r, b, m_1 and m_2, a_1 and a_2, and y_0. y_0 is the maximum or mode of the theoretical frequency curve. Constant a_1 is the distance from the mode to the lower limit, while a_2 is the distance to the upper limit. The expressions for these constants are as follows:

$$r = \frac{6(B_2-B_1-1)}{6+3B_1-2B_2}$$

$$b = \tfrac{1}{2}\sqrt{\mu_2}\sqrt{B_1(r+2)^2+16(r+1)}$$

$$m_2 \text{ and } m_1 = \tfrac{1}{2}\left[r - 2 \pm r(r+2)\sqrt{\frac{B_1}{B_1(r+2)^2 + 16(r+1)}}\,\right]$$

$$a_1 + a_2 = b \text{ and } \frac{a_1}{m_1} = \frac{a_2}{m_2}$$

$$y_0 = \frac{N m_1^{m_1} m_2^{m_2}\,\Gamma(m_1 + m_2 + 2)}{b(m_1 + m_2)^{m_1 + m_2}\,\Gamma(m_1 + 1)\,\Gamma(m_2 + 1)}$$

The substituted values for frequency group III are as follows:

$$r = \frac{6(2.7294 - 0.00558 - 1)}{6 + 3(0.00558) - 2(2.7294)} = 18.541$$

$$b = \tfrac{1}{2}\sqrt{6.7192}\sqrt{0.00558(20.541)^2 + 16(19.541)} = 23.0014$$

$$m_2 \text{ and } m_1 = \tfrac{1}{2}\left[16.541 \pm 18.541(20.541)\sqrt{\frac{0.00558}{315.01}}\right]$$

$$= \tfrac{1}{2}(16.541 \pm 1.503)$$

$$m_1 = \tfrac{1}{2}(18.044) = 9.022$$

$$m_2 = \tfrac{1}{2}(15.038) = 7.519$$

$$a_1 + a_2 = b = 23.0014$$

$$\frac{a_1}{9.022} = \frac{a_2}{7.519} \qquad a_1 = 12.5457 \qquad a_2 = 10.4557$$

$$y_0 = \frac{100(9.022)^{9.022}(7.519)^{7.519}\,\Gamma\,18.541}{23.0014(16.541)^{16.541}\,\Gamma\,10.022\,\Gamma\,8.519} = 14.79.$$

In a similar manner the constants were calculated for the remaining frequency groups. The values of these constants are tabulated in Table 26 for convenient comparison. Figure 46 shows the variation of these constants from Types I to VII. All of the constants follow some tendency in values over the range of types. The tendencies are especially striking for constants r, m_1, a_1, a_2, and y_0.

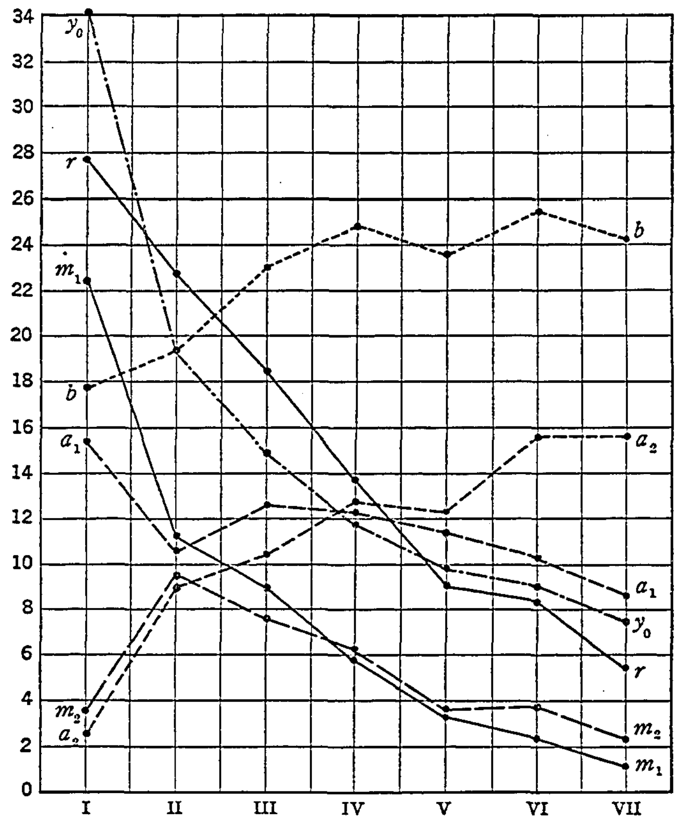


Figure 46. Variation of Frequency Equation Constants with Types

Table 26. Frequency Equation Constants for Seven Frequency Groups

Type	r	b	m_1	m_2	a_1	a_2	y_0
I..........	27.875	17.821	22.474	3.400	15.474	2.342	34.15
II..........	22.6036	19.3897	11.1285	9.4751	10.4728	8.917	19.28
III..........	18.541	23.0014	9.022	7.519	12.5457	10.4557	14.79
IV..........	13.883	24.747	5.824	6.059	12.129	12.618	11.82
V..........	9.0192	23.651	3.387	3.632	11.413	12.238	9.86
VI..........	8.3268	25.622	2.4995	3.8272	10.1225	15.4995	8.90
VII..........	5.3585	24.0966	1.1986	2.1599	8.5997	15.4969	7.574

Calculating Ordinates from Frequency Curve Equations.—In calculating ordinates from the frequency equations, the change in abscissa scale must be taken account of. The values of x used in the calculation of moments were measured from the mean; but the x's in the equations are distances from the mode. The position of the mode with respect to the mean can, however, be determined from the following expression:

$$\text{Mode} = \text{Mean} - \tfrac{1}{2}\left[\frac{\mu_3\,(r+2)}{\mu_2\,(r-2)}\right]$$

In substituting the values for μ_1, μ_2, and r for group III in this expression, the position of the mode becomes known.

$$\text{Mode} = \text{Mean} - \tfrac{1}{2}\,\frac{(-\,1.3012)\,(18.542+2)}{(6.7192)\,(18.542-2)} = +\,0.12024$$

Table 27 gives the position of the mode with respect to the mean for each of the seven frequency groups as found by the use of above equation.

Table 27. Position of Mode with Respect to Mean for Seven Frequency Groups

Type	Mode = Mean ±
I	+ 0.47133
II	+ 0.218156
III	+ 0.12024
IV	− 0.003526
V	− 0.1178
VI	− 0.63754
VII	− 1.2877

In calculating the values of the ordinates from the frequency equation it is of advantage to be able to compare the values of ordinates with the values of x used in the original data. This is accomplished by using for the value of x of a given point measured from the mean, the distance on the x scale from the same point to the mode. As an example, let it be assumed that the ordinate at the point $x = -4\tfrac{1}{2}$ measured from the mean is desired. This value of $x = -4\frac{1}{2}$ is -4.62024 units from the mode. (See Figure 47.) Therefore, by using -4.62024 for x in the equation, the desired value of y can be found. This plan of substitution for x is used throughout in the calculations for y in the seven equations.

Table 28 illustrates the steps in the calculations for the frequency ordinates. Column (1) gives the values of x measured from the mean, and column (1a) the corresponding values measured from the mode. Columns (2), (3), (4), (5), (6),

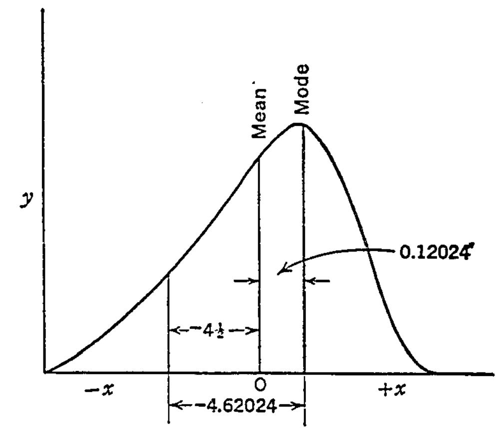


Figure 47. Relation of Mode to Mean in Frequency Group III

(7), and (8) are steps in the determination of the logarithm of y_x. Column (10) gives the values of y_x, and column (11) the values of y from the original frequency curve from which the equation was derived.

Comparison with Original Frequency Curves.—The values of y_x as calculated from the seven frequency equations are tabulated in Table 29. In adjacent columns the values for y from the original seven type frequency curves are also given. A comparison of the values as obtained from the equations with the original data shows that satisfactory agreement was obtained. The closeness of fit is still better shown by a comparison of the corresponding frequency curves. These are shown

Table 28. Steps in the Calculation of the Ordinates from the Frequency Equation for Group III

(1) x	(1a) $x + (0.12024)$	(2) $1 + \frac{x}{12.5457}$	(3) $1 - \frac{x}{10.4557}$	(4) log (2)	(5) log (3)	(6) $9.022 \times (4)$	(7) $7.519 \times (5)$	(8) $6 + 7 + \log y_0$	(9) y_x	(10) y
−9½	9.62024									
−8½	8.62024	0.3129	1.8244	$\bar{1}.49541$	0.26109	−4.55241	1.96314	$\bar{2}.58070$	0.04	
−7½	7.62024	0.3926	1.7288	$\bar{1}.59395$	0.23774	−3.66338	1.78757	$\bar{1}.29416$	0.20	0.30
−6½	6.62024	0.4723	1.6332	$\bar{1}.67422$	0.21304	−2.93919	1.60185	$\bar{1}.83263$	0.68	0.90
−5½	5.62024	0.5520	1.5375	$\bar{1}.74194$	0.18682	−2.33724	1.40470	0.23743	1.73	1.80
−4½	4.62024	0.6317	1.4419	$\bar{1}.80051$	0.15894	−1.79980	1.19507	0.56524	3.70	3.00
−3½	3.62024	0.7114	1.3462	$\bar{1}.85211$	0.12911	−1.33426	0.97078	0.80649	6.42	5.80
−2½	2.62024	0.7912	1.2506	$\bar{1}.89829$	0.09709	−0.91763	0.73002	0.98236	9.62	10.30
−1½	1.62024	0.8709	1.1550	$\bar{1}.93997$	0.06258	−0.54159	0.47054	1.09892	12.58	14.10
− ½	0.62024	0.9506	1.0593	$\bar{1}.97800$	0.02502	−0.19848	0.18812	1.15961	14.36	13.90
½	0.37976	1.0303	0.9637	0.01296	$\bar{1}.98394$	0.11693	−0.12075	1.16115	14.51	13.80
1½	1.37976	1.1099	0.8680	0.04528	$\bar{1}.93852$	0.40852	−0.46227	1.11622	13.09	13.20
2½	2.37976	1.1897	0.7724	0.07544	$\bar{1}.88784$	0.68062	−0.84333	1.00726	10.19	10.40
3½	3.37976	1.2694	0.6768	0.10359	$\bar{1}.83046$	0.93459	−1.27477	0.82979	6.78	6.20
4½	4.37976	1.3491	0.5811	0.13004	$\bar{1}.76425$	1.17322	−1.77260	0.57059	3.74	3.70
5½	5.37976	1.4288	0.4855	0.15497	$\bar{1}.68619$	1.39814	−2.35954	0.20857	1.64	2.20
6½	6.37976	1.5085	0.3898	0.17854	$\bar{1}.59084$	1.61079	−3.07647	$\bar{1}.70429$	0.52	0.40
7½	7.37976	1.5882	0.2942	0.20091	$\bar{1}.46864$	1.81261	−3.99529	$\bar{2}.98729$	0.10	
8½	8.37976									
9½	9.37976									

Table 29. Comparison of Original Replacement Data of Seven Type Frequency Curves with Values Obtained from Frequency Equations

[Original Replacement Data in Columns (1), (3), (5), (7), (9), (11), and (13). Values from Equations in Columns (2), (4), (6), (8), (10), (12), and (14).]

Age Interval	Types													
	I		II		III		IV		V		VI		VII	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
0– 10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.50	0.22	1.40	0.26	1.60	0.46
10– 20	0.00	0.00	0.00	0.00	0.00	0.04	0.50	0.24	1.00	0.74	1.20	1.01	1.80	1.94
20– 30	0.00	0.00	0.00	0.00	0.30	0.20	1.20	1.00	1.40	1.61	1.90	2.14	2.40	3.38
30– 40	0.00	0.00	0.00	0.01	0.90	0.68	1.40	1.68	1.90	2.82	2.30	3.49	3.10	4.64
40– 50	0.00	0.05	0.60	0.32	1.80	1.73	2.40	3.14	3.00	4.27	3.50	4.93	5.60	5.68
50– 60	0.00	0.27	2.10	1.38	3.00	3.70	3.90	3.04	6.10	5.81	7.00	6.28	6.90	6.50
60– 70	2.40	1.27	4.10	4.02	5.70	6.42	7.70	7.17	8.50	7.28	8.70	7.43	8.00	7.07
70– 80	4.50	4.61	7.80	8.53	10.30	9.62	11.50	9.22	10.00	8.50	9.00	8.28	7.60	7.43
80– 90	11.10	12.75	15.30	14.02	14.10	12.58	11.50	10.81	9.80	9.39	8.80	8.77	7.50	7.56
90–100	27.00	25.96	19.10	18.34	13.90	14.46	10.70	11.70	8.80	9.82	7.70	8.91	7.20	7.51
100–110	35.50	34.16	19.50	19.24	13.80	14.51	10.70	11.70	9.40	9.77	8.50	8.69	7.30	7.28
110–120	16.50	20.34	16.50	16.13	13.20	13.09	10.30	10.83	8.60	9.22	7.60	8.15	7.00	6.91
120–130	3.00	0.59	9.00	10.59	10.40	10.19	9.70	9.24	8.50	8.27	7.60	7.24	7.60	6.40
130–140			4.60	5.24	6.30	6.78	7.40	7.23	6.80	7.02	6.70	6.40	5.60	5.80
140–150			1.40	1.71	3.70	3.74	4.70	5.13	5.60	5.58	5.00	5.32	4.90	5.11
150–160				0.37	2.20	1.64	2.80	3.25	4.00	4.11	3.90	4.23	4.40	4.38
160–170				0.10	0.40	0.52	1.90	1.80	2.40	2.75	3.20	3.19	3.60	3.63
170–180						0.10	1.70	0.82	2.40	1.62	2.50	2.26	2.40	2.89
180–190									1.39	0.80	1.70	1.47	2.40	2.18
190–200										0.34	1.30	0.87	1.30	1.53
200–210										0.06	0.50	0.44	1.30	0.97
210–220												0.18	0.50	0.52
220–230												0.06		0.20
230–240														0.02

Table 30. Comparison of Per Cent Survivors of Seven Type Mortality Curves with Per Cent Survivors Obtained from Frequency Equations

[Original Survivor Data in Columns (1), (3), (5), (7), (9), (11), and (13). Values from Equations in Columns (2), (4), (6), (8), (10), (12), and (14).]

Age Interval	Types													
	I		II		III		IV		V		VI		VII	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
0– 10	100.0	100.00	100.0	100.00	100.0	100.00	100.0	100.00	100.0	100.00	100.0	100.00	100.0	100.00
10– 20	100.0	100.00	100.0	100.00	100.0	100.00	100.0	100.00	99.5	99.78	98.6	99.74	98.4	99.54
20– 30	100.0	100.00	100.0	100.00	100.0	99.96	99.5	99.76	98.5	99.04	97.4	98.73	96.6	97.60
30– 40	100.0	100.00	100.0	100.00	99.7	99.76	98.3	98.76	97.1	97.43	95.5	96.59	94.2	94.22
40– 50	100.0	100.00	100.0	99.99	98.8	99.08	96.9	97.08	95.2	94.61	93.2	93.10	91.1	89.58
50– 60	100.0	99.95	99.4	99.67	97.0	97.35	94.5	93.94	92.2	90.34	89.7	88.17	85.5	83.90
60– 70	100.0	99.68	97.3	98.29	94.0	93.65	90.6	88.90	86.1	84.53	82.7	81.89	78.6	77.40
70– 80	97.6	98.41	93.2	94.27	88.3	87.23	82.9	81.73	77.6	77.25	74.0	74.46	70.6	70.33
80– 90	93.1	93.80	85.4	85.74	78.0	77.61	71.4	72.51	67.6	68.75	65.0	66.18	63.0	62.90
90–100	82.0	81.05	70.1	71.72	63.9	65.03	59.9	61.70	57.8	59.36	56.2	57.41	55.5	55.34
100–110	55.0	55.09	51.0	53.38	50.0	50.57	49.2	50.00	49.0	49.54	48.5	48.50	48.3	47.82
110–120	19.5	20.93	31.5	34.14	36.2	36.06	34.5	38.30	39.6	39.77	40.0	39.81	41.0	40.54
120–130	3.0	0.59	15.0	18.01	23.0	22.97	28.2	27.42	31.0	30.55	32.4	31.66	34.0	33.63
130–140	0.0	0.00	6.0	7.42	12.6	12.78	18.5	18.23	22.5	22.28	24.8	24.42	26.4	27.23
140–150			1.4	2.18	6.3	6.00	11.1	11.00	15.7	15.26	18.1	18.02	20.8	21.43
150–160			0.0	0.47	2.6	2.26	6.4	5.87	10.1	9.68	13.1	12.70	15.9	16.32
160–170				0.10	0.4	0.62	3.6	2.62	6.1	5.57	9.2	8.47	11.5	11.94
170–180				0.00	0.0	0.10	1.7	0.82	3.7	2.82	6.0	5.28	7.9	8.31
180–190						0.00	0.0	0.00	1.3	1.20	3.5	3.02	5.5	5.42
190–200									0.0	0.40	1.8	1.55	3.1	3.24
200–210										0.06	0.5	0.68	1.8	1.71
210–220										0.00	0.0	0.24	0.5	0.74
220–230												0.06	0.0	0.22
230–240												0.00		0.02
240–250														0.00

in Figures 48 to 54. That a satisfactory fit was obtained in each case is apparent; in fact, the agreement is remarkable.

Comparison with Original Mortality Curves.—To show to what extent the original seven type mortality curves vary from the survivor data obtained from the frequency equations,

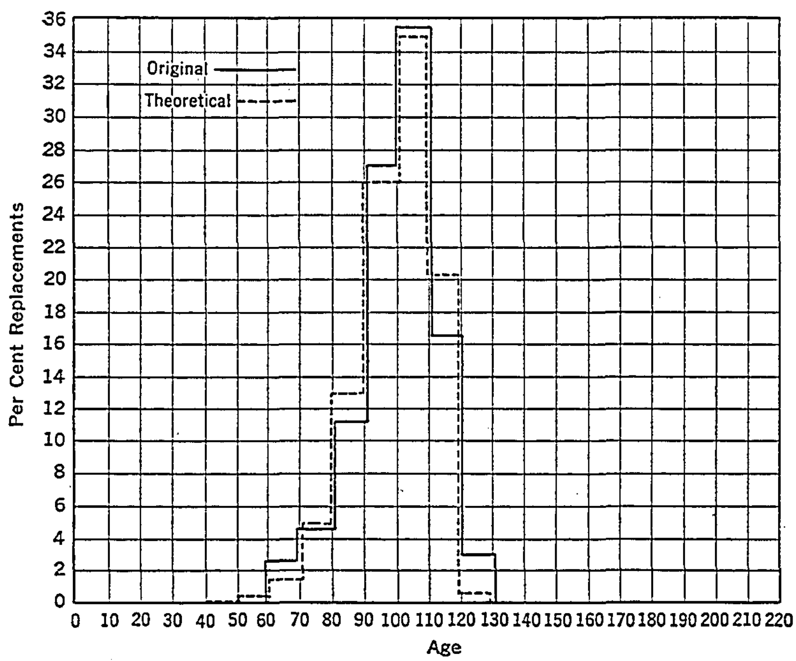


Figure 48. Comparison of Original and Theoretical Frequency Curves for Type I

$$y = 34.15\left(1 + \frac{x}{15.479}\right)^{22.474}\left(1 - \frac{x}{2.342}\right)^{3.400}$$

$$\text{Mode} = \text{Mean} + 0.47133$$

Table 30 has been prepared. In this table are given the per cent survivors for the seven original type mortality curves, and the per cent survivors obtained from the replacement data calculated from the frequency equations and tabulated in Table 29. The variations in the percentages can be better noted by com-

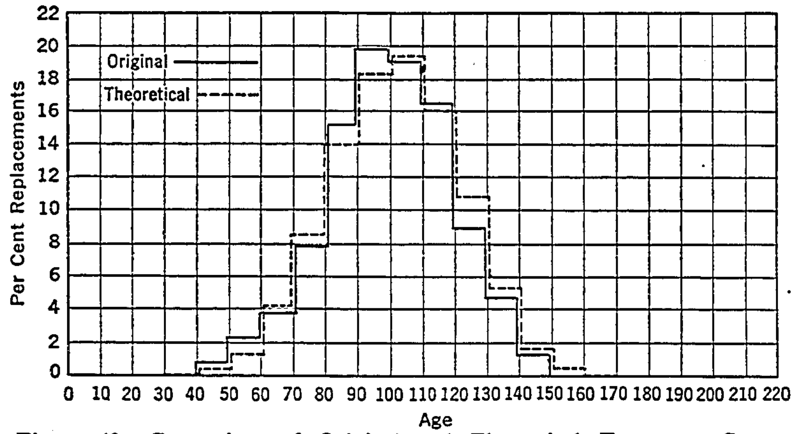


Figure 49. Comparison of Original and Theoretical Frequency Curves for Type II

$$y = 19.285\left(1 + \frac{x}{10.47284}\right)^{11.1285}\left(1 - \frac{x}{8.9196}\right)^{9.4751}$$

Mode = Mean + 0.218156

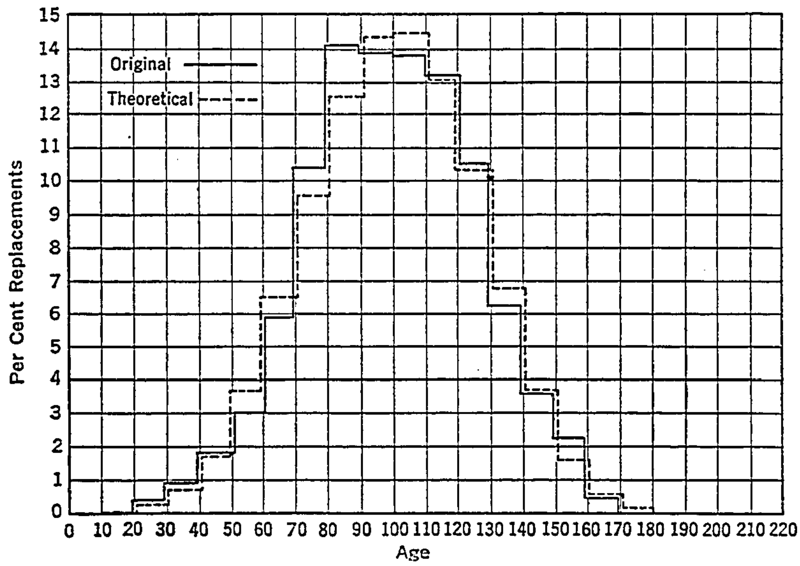


Figure 50. Comparison of Original and Theoretical Frequency Curves for Type III

$$y = 14.79\left(1 + \frac{x}{12.5457}\right)^{9.022}\left(1 - \frac{x}{10.4557}\right)^{7.519}$$

Mode = Mean + 0.12024

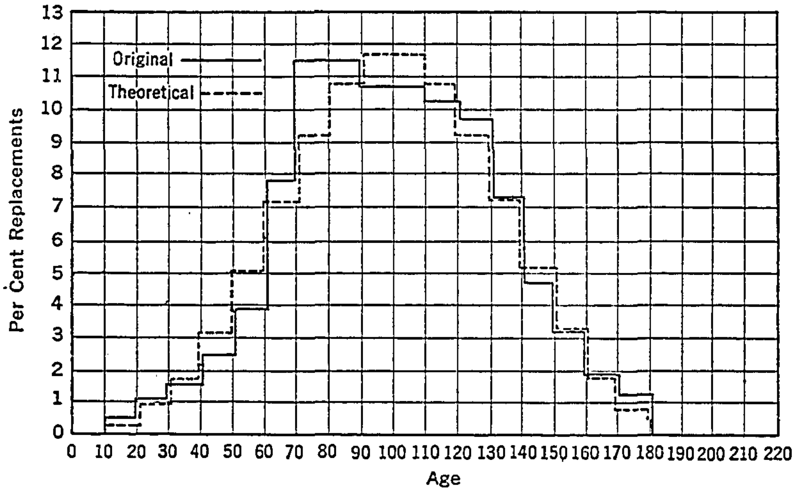


Figure 51. Comparison of Original and Theoretical Frequency Curves for Type IV

$$y = 11.82\left(1 + \frac{x}{12.618}\right)^{6.059}\left(1 - \frac{x}{12.129}\right)^{5.824}$$

Mode = Mean − 0.003526

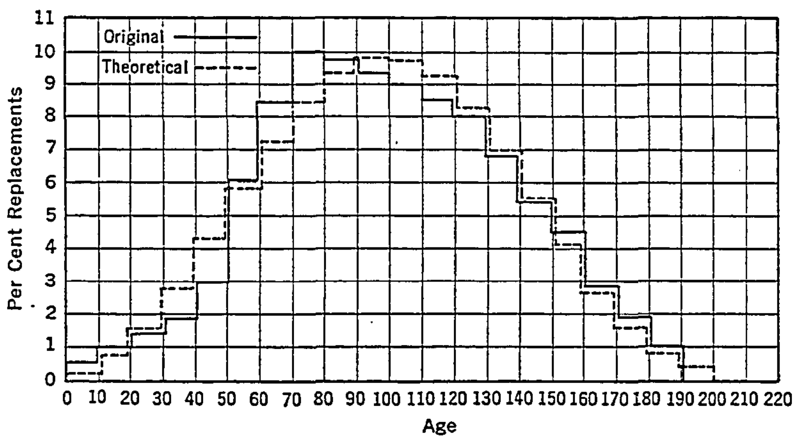


Figure 52. Comparison of Original and Theoretical Frequency Curves for Type V

$$y = 9.86\left(1 + \frac{x}{11.413}\right)^{3.337}\left(1 - \frac{x}{12.238}\right)^{3.632}$$

Mode = Mean − 0.1178

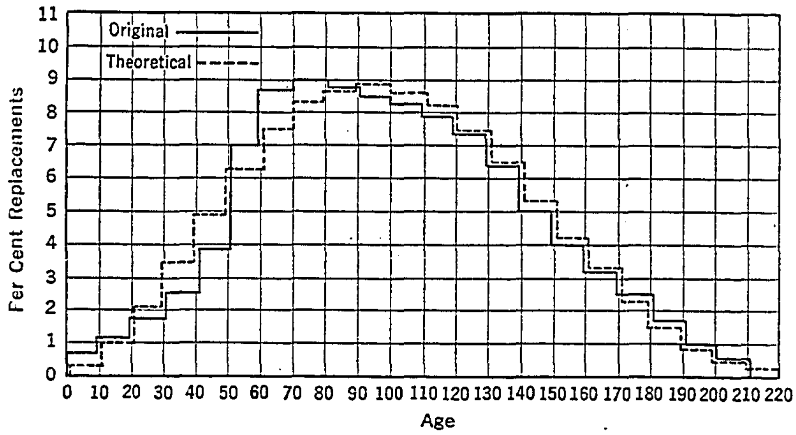


Figure 53. Comparison of Original and Theoretical Frequency Curves for Type VI

$$y = 8.90\left(1+\frac{x}{10.1225}\right)^{2.4995}\left(1-\frac{x}{15.4995}\right)^{3.8272}$$

Mode = Mean − 0.63754

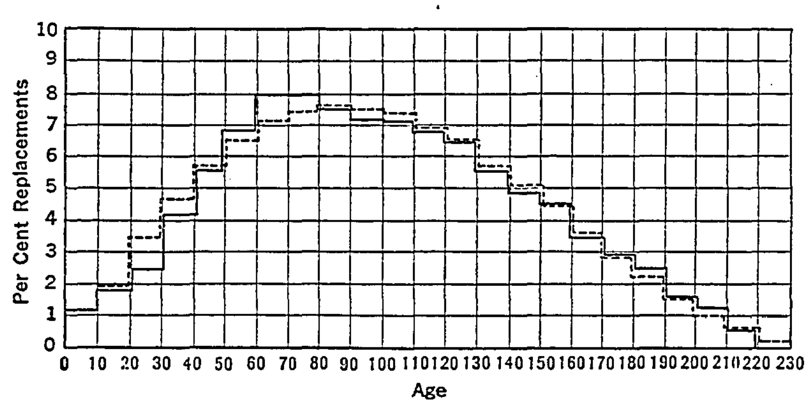


Figure 54. Comparison of Original and Theoretical Frequency Curves for Type VII

$$y = 7.574\left(1+\frac{x}{8.5997}\right)^{1.1986}\left(1-\frac{x}{15.4969}\right)^{2.1599}$$

Mode = Mean − 1.2877

paring the plotted values. Figure 55 shows the seven original type mortality curves and the plotted points of the data obtained from the frequency equations. The closeness of fit is equally as striking as it was in the case of the frequency curves themselves. It is obvious that no serious concession would be

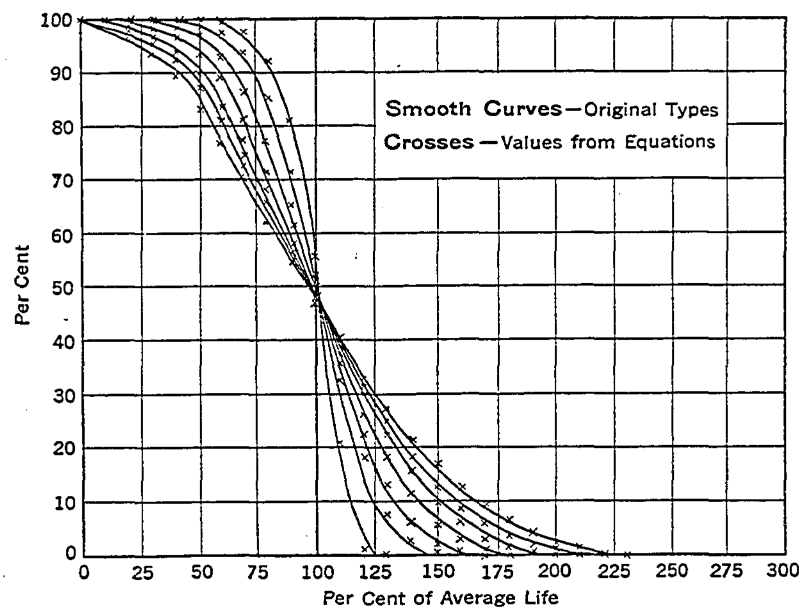


Figure 55. Comparison of Original Seven Type Survivor Curves with Values of Per Cent Survivors Obtained from Theoretical Frequency Equations

made to allow the curves obtained from the frequency equations to become the seven type curves.

Significance of Frequency Equations.—The significance of the seven frequency equations is twofold:

First, the equations provide a means for graduating the seven type frequency curves. By their use irregularities are smoothed out and a more reliable curve results. Moreover, since the mortality curves are derived from the frequency curves, the mortality curves will be graduated likewise. Studies of the seven type mortality or frequency curves made hereinafter will therefore be based on those obtained from the frequency equations.

Second, the fact that the equations fit the original data remarkably well establishes the fact that these observations regarding the life of physical property can be classed as natural phenomena. This is a very important conclusion and will be further developed in later chapters.

Relation of Frequency Types to Kinds of Property.— The 52 mortality curves were classified into the seven type mortality curves in Table 20, Chapter 3. The classification into frequency curve types naturally is the same, as the seven frequency curve types were obtained from the corresponding seven mortality curve types. The classes of property, like waterworks, telephone, telegraph, and electric, fall largely into Types VI and VII with a few in the middle types, V, IV, and III. These frequency curve types undergo continuous replacements from the beginning to the end of life; in fact, Type VII has removals over a period of 220% of average life. At no time, however, are the replacements very large, reaching a maximum of only 8% in Type VII. Railroad equipment and ties on the other hand fall largely into the lower classes, I, II, and III, indicating that the renewals or replacements take place over a much shorter period of time, and at the same time reach higher values per interval. As already noted in Type I, the replacements reach a peak of over 35% per interval but are only spread over a period equal to 70% of average life.

CHAPTER 5

EQUATIONS OF THE SEVEN TYPE MORTALITY CURVES

DeMoivre's Hypothesis.—The first recorded analysis of life tables of humans was made by DeMoivre in 1725. At that time only a limited number of mortality tables were in existence, and the methods in use of computing tables were clumsy and tedious. Any scheme that would save time and labor in the actuarial calculations was therefore much desired. It was with this objective in mind that DeMoivre began the analysis of life tables, which resulted in his conception of the hypothesis of "equal decrements." This hypothesis was published in his "Treatise of Annuities on Lives" in 1725. His assumption was simply that the values l_x in the survivor column decreased uniformly up to the limiting age. The hypothesis thus consists in supposing the values in the survivor column to decrease in an arithmetical progression, and that to fit the tables then in existence the hypothesis had to be confined to the period of life between ages 12 and 86. The number living at age 12 could thus be assumed to be $86 - 12 = 74$, and the survivors at each age thereafter are obtained by successively decreasing this value by one, thus 73, 72, 71, 70, etc. These survivors correspond to ages, 13, 14, 15, 16, 17, etc., respectively. At age 86 the arithmetical progression reaches 0, and the end of the life table is reached.

A cursory examination of the seven type curves shows that DeMoivre's hypothesis was indeed only a very rough approximation of the present-day tables of physical property. The hypothesis assumes a straight horizontal line for a distribution curve, whereas an inspection of these curves in Chapter 4 shows them to be skew curves. Since DeMoivre's time, many more mortality tables of humans have become available and further study has been given this interesting subject with the primary object of discovering the law of mortality, rather than to gain facilities for making numerical calculations. These studies have resulted in formulas and mathematical expressions which, though they may not absolutely embody the law of mortality, seem very suggestive.

Gompertz's Geometrical Progression.—In 1825, 100 years after the conception of the hypothesis of "equal decrements" by DeMoivre, Benjamin Gompertz contributed a paper to the Royal Society in which he discussed the effects of supposing the survivors at successive ages to be in geometrical progression instead of arithmetical progression. In Article 4 of this paper he says:

"It is possible that death may be the consequence of two generally coexisting causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or increased inability to withstand destruction. If, for instance, there be a number of diseases to which the young and old are equally liable, and likewise which should be equally destructive whether the patient be young or old, it is evident that the deaths among the young and old by such diseases would be exactly in proportion of the number of young to the old; provided those numbers were sufficiently great for chance to have its play; and the intensity of mortality might then be said to be constant; and were there no other diseases but such as those, life of all ages would be of equal value, and the number of living and dying, from a certain number living at a given earlier age, would decrease in geometrical progression, as the age increased by equal intervals of time; but if mankind be continually gaining seeds of indisposition, or in other words, an increased liability to death (which appears not to be an unlikely supposition with respect to a great part of life, though the contrary appears to take place at certain periods), it would follow that the number of living out of a given number of persons at a given age, at equal successive increments of age, would decrease in a greater ratio than the geometrical progression, and then the chances against the knowledge of any one having arrived to certain defined terms of old age might increase in a much faster progression, notwithstanding there might still be no limit to the age of man."

Briefly, this hypothesis may be stated by saying that the rate of change of the force of mortality (called intensity of mortality by Mr. Gompertz) at any time is proportional to the force of mortality. That is, calling μ_x the force of mortality,

$$\frac{d\mu_x}{dx} = k\mu_x$$

Transposing,
$$\frac{d\mu_x}{\mu_x} = kdx$$

Integrating
$$\log \mu_x = kx + \text{constant of integration}$$

or
$$\mu_x = Be^{kx} = Bc^x$$

where B and c are constants.

Now, the force of mortality μ_x at any age may be defined as the rate of change of the average number living l_x of an indefinitely large class of persons of age x, per individual of the group l_x. That is to say, if the function l_x has a derivative with regard to x, then,

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$$

In other words, l_x is taken to be a continuous function proportional to the average number living at age x out of an indefinitely large group of lives of age in the neighborhood of x. Then the force of mortality is defined as the derivative of this function with regard to age or time divided by l_x. Obviously,

l_x is a decreasing function with time. The negative sign is placed before the derivative $\frac{dl_x}{dx}$ so that μ_x is positive.

Using this expression for μ_x and substituting in the equation for Gompertz's hypothesis, gives,

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = Bc^x$$

Integrating, and the usual form of Gompertz's law is obtained, thus,

$$-\frac{1}{l_x}\frac{dl_x}{dx} = Bc^x$$

$$-\frac{dl_x}{l_x} = Bc^x dx$$

$$-\log l_x = Bc^x + \text{integration constant}$$

or

$$l_x = Be^{c^x} + \text{constant}$$

$$= kg^{c^x}$$

where k, g, and c are constants to be evaluated for each mortality table.

In the examples which accompanied the first publication of this formula, Gompertz restricted the use of the formula to the period between 15 and 55 years.

Makeham's Modification of the Gompertz Formula.—In 1860, Mr. Makeham published a paper in the *Journal of the Institute of Actuaries* in which he discussed Gompertz's formula for the law of human mortality and introduced his first modification thereof, which greatly improved its accuracy and increased its range. The hypothesis back of this modification is that the force of mortality is given by,

$$\mu_x = A + Bc^x \text{ instead of } Bc^x \text{ only.}$$

That is, the force of mortality consists of two parts: the one part increases in geometrical progression with age, while the other is constant. Equating again the expression for μ_x, the force of mortality, and the expression for μ_x as given above, gives,

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = A + Bc^x$$

or

$$-\frac{dl_x}{l_x} = Adx + Bc^x dx$$

Integrating,

$$-\log l_x = Ax + Bc^x + \text{constant}$$

or

$$l_x = ks^x g^{c^x}$$

This modification was, in fact, suggested by Mr. Gompertz. himself, when he said that it was possible that death might be the consequence of two coexisting causes, namely, one, chance, without any previous disposition to death, the other, an increased inability to withstand destruction; and that as regards the first, the intensity of mortality could be assumed constant; while as regards the second the intensity would be an increasing geometrical progression. In deducing his formula, however, he only gave consideration to the second of these causes of death and neglected to include the first. If he had included the first he might have arrived at the same conclusion as Makeham.

Since the formula for the law of human mortality is the result of the work of Gompertz and Makeham, it is sometimes referred to as the Gompertz-Makeham formula. This formula is widely employed at the present time by actuaries of life insurance companies in the graduation of human life tables upon which these companies base premium rates for the sale of life insurance and annuity contracts.

Procedure in Determination of Formula Constants.—A number of methods for the determination of the values of the constants c, g, s, and k, have been developed. Descriptions of several methods are presented in the Institute of Actuaries' Textbook by King, pages 74-104. The author has used a number of these, but has secured the best results by the use of a graphical method. This method is of interest, first, because it involves an original idea, and second, because it demonstrates convincingly that although the Gompertz-Makeham equation can be applied in form to physical property, it does not apply in principle. The writer considers this an outstanding discovery in the field of physical property mortality laws, and almost as significant as the contribution made by Makeham in 1860.

In attempting to fit equations to experimental or observational data it will often be found that no single, simple expression will be satisfactory. The use of a single, simple expression may fit one part of the curve quite well but not fit the other parts at all. In such cases a modification of the single, simple expression by the addition of one or more terms will sometimes cause the equation to fit the curve approximately throughout its entire length. An illustration of such a modified equation is the following:

$$y = a + bx + ce^{dx}$$

The first part of the equation, "$a + bx$," is the simple expression and is the equation of a straight line with intercept on the y axis of a and slope b. The deviation of this straight line from the remainder of the curve, however, must be compensated or corrected for. This correction may be secured by the addition of one or more terms, which frequently have the form ce^{dx} or cd^x, making the whole equation appear as follows:

$$y = a + bx + cd^x$$

The deviation of the straight line from the experimental or observational curve is found by calculating the residuals, thus,

$$r = y_x - (a + bx)$$

where r is the residual, $a + bx$ the equation of the straight line, and y_x the original data corresponding to the values of x. The nature of the deviations is studied by plotting values of r against x. If this plot is a simple exponential curve the term to be added to provide for the correction is of the form ce^{dx}, with the values of the constants c and d such that the value of the term is negligible for that part of the curve to which the straight line $a + bx$ has been fitted. In case no part of the curve is a straight line the form of the equation given may still be used, but then the term ce^{dx} must not be negligible for any values of x.

The Gompertz-Makeham formula can be put in the form of the equation discussed above by writing the formula in logarithmic terms, thus:

$$y = ks^x g^{c^x}$$
$$Y = \log y = \log k + (\log s)x + (\log g)c^x$$

This equation is of the same form as

$$y = a + bx + cd^x$$

When y is replaced by $\log y$
a is replaced by $\log k$
b is replaced by $\log s$
and c is replaced by $\log g$

The first step in the determination of the values of the constants k, s, g, and c, consists in obtaining the logarithms of the values of l_x, and then plotting these values of $\log l_x$ against x. Table 31 gives the values of l_x and $\log l_x$ for Type V of the seven type mortality curves, which will be used for illustration of the procedure. Figure 56 shows the values of $\log l_x$ plotted against x. As no part of this curve is a straight line the terms $\log k + x(\log s)$ cannot be fitted to the curve itself. This is not essential, however, as another requirement must be fulfilled, namely, the logarithms of the residuals must form a straight line when plotted against x. The secret, therefore, lies in so locating the straight line represented by $\log k + x(\log s)$ that the logarithms of the residuals will lie along a straight line. This was the novel feature introduced in this graphical method. The straight line which produced the desired result is, $Y' = 2.100 + 0.02x$, where 2.100 is the intercept, and 0.02 the positive slope.

Table 31. Logarithms of Survivors and Residuals for Type V Mortality Curve

x (1)	l_x (2)	$\log l_x$ (3)	z (4)	$\log z$ (5)	
0	100.0	2.00000	0.10	$\bar{1}.000$	-1.00000
1	99.78	1.99904	0.12	$\bar{1}.07918$	-0.92082
2	99.04	1.99581	0.145	$\bar{1}.16137$	-0.83863
3	97.43	1.98869	0.17	$\bar{1}.23045$	-0.76955
4	94.61	1.97594	0.205	$\bar{1}.31175$	-0.68825
5	90.34	1.95588	0.245	$\bar{1}.38917$	-0.61083
6	84.53	1.92701	0.295	$\bar{1}.46982$	-0.53018
7	77.25	1.88790	0.355	$\bar{1}.55023$	-0.44977
8	68.75	1.83727	0.42	$\bar{1}.62325$	-0.37675
9	59.36	1.77349	0.50	$\bar{1}.69897$	-0.30103
10	49.54	1.69496	0.60	$\bar{1}.77815$	-0.22185
11	39.77	1.59956	0.72	$\bar{1}.85733$	-0.14267
12	30.55	1.48501	0.86	$\bar{1}.93450$	-0.06550
13	22.28	1.34792	1.01	0.00432	
14	15.26	1.18355	1.20	0.07918	
15	9.68	0.98588	1.41	0.14922	
16	5.57	0.74586	1.68	0.22531	
17	2.82	0.45025	1.99	0.29885	
18	1.20	0.07918			
19	0.40				

The second step consists in subtracting this expression for Y' from the expression for Y, thus:

$$\begin{array}{lcl} Y & = & \log k + x(\log s) + (\log g)c^x \\ Y' & = & \log k + x(\log s) \\ \hline Z = Y - Y' & = & (\log g)c^x \end{array}$$

In this expression, $Z = (\log g)\ c^x$, the quantity $(\log g)$ will always be negative, because the correction is one of subtraction, and c will always be greater than 1 because the correction must increase with increasing values of x. The values of Z may be found by evaluating Y' for each value of x, and subtracting from Y. Approximate values of Z can also be obtained by stepping off the vertical distances between the straight line representing Y' and the curve for Y. The values for Z for Type V are also given in Table 31 in column (4).

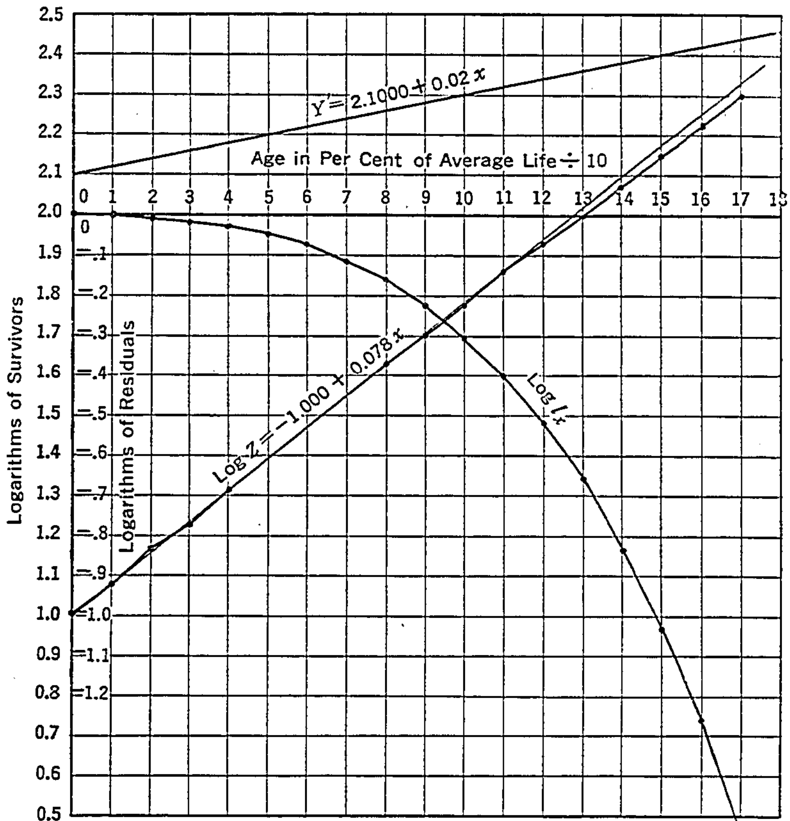


Figure 56. Method of Determining Constants in Makeham-Gompertz Formula for Type V Mortality Curve

The next step consists in taking the logarithms of both sides of the expression $Z = (\log g)\ c^x$, thus:

$$\log Z = \log (\log g) + x(\log c)$$

It is noted that this is the equation of another straight line with log (log g) as intercept, and (log c) as slope. By obtaining the logarithms of the residuals Z, given in column (4) of Table

31, and plotting against x, the line shown in Figure 56 results. The points fall on an approximately straight line as the position of the line $\log k + x(\log s)$ was selected to secure this result. This line can be represented by the expression:

$$\log Z = -1.000 + 0.078x$$

in which —1.000 is the intercept and 0.078 the slope. From this:

$$Z = -0.1000(1.197)^x$$

The last step consists in adding the two components of the expression for Y, thus,

$$Y = Y' + Z$$
$$Y = 2.100 + 0.02x - 0.1000(1.197)^x,$$

of which the antilogarithms are:

$$y = 125.9 \times 1.047^x \times 0.7943^{1.197^x}$$

The value of the constants for the other six type mortality curves was determined in like manner.

Constants and Equations for Seven Type Mortality Curves.—The constants, as obtained by the graphical method described in the preceding paragraph, for the seven type mortality curves are tabulated in Table 32. As already noted, the logarithm of g is always negative. In accordance with Makeham's analysis in 1860, as well as Gompertz's statement in 1825, the logarithm of s should also be negative as both should tend to reduce the number of survivors with increase of age. Makeham's analysis showed that the logarithm of the probability of living t years could be more accurately represented if a negative term be added to the geometrical progression. In explanation of this it was pointed out that death was likely due to two generally coexisting causes; the one, chance without any previous disposition to death; the other, a deterioration or increased inability to withstand destruction. The intensity of the first cause was assumed constant while the second would be an increasing geometrical progression.

The analysis of the equations in this chapter show conclusively that the quantity s must be positive instead of negative to obtain the best fit. Therefore, although the form of the Gompertz-Makeham formula can still be used, it does not hold in principle. The effect of the constant s, in fact, is directly opposite in its action. This effect is, of course, partly compensated for by larger values of g and c. Nevertheless, the reversal of the sign of s is interesting, and is considered a fundamental discovery in the realm of physical property mortality laws.

The manner in which the constants vary over the range of types is shown in Figure 57. The log g starts with almost zero value for Type I and increases almost uniformly to Type VII. Constant c, however, starts with its highest value, 2.83, and decreases rapidly at first to 1.291 in Type III, and then very slowly to 1.137 for Type VII. These values of c are much higher than are common for human mortality tables. In the case of humans the value of c is usually around 1.10.

The equations embodying these constants are given in Table 33 both in the logarithmic and natural form.

Comparison of Values from Equations With Data.—In order to show how well the seven equations represent the data from which they were derived, Figures 58 to 64 inclusive have been prepared. These figures each show a smooth curve, located by points, which is the survivor curve of the data obtained from the frequency equations. In addition, each figure contains a number of crosses which represent values computed from the equation by substituting given values of x. It is evident that the agreement is very satisfactory.

It seems that the form of the Gompertz-Makeham formula applies even better to physical property than it does to humans. This is due largely to the fact that there is no infant mortality

Table 32. Values of Constants of Gompertz-Makeham Formula for Seven Type Mortality Curves

Type	k		s		g		c	
	log	antilog	log	antilog	log	antilog	log	antilog
I	2.0000073	100.0	0.00	1.00	− 0.000007291	0.9999	0.4518	2.83
II	2.000204	100.05	0.00	1.00	− 0.0002042	0.9995	0.305	2.018
III	2.04170	110.1	0.02	1.047	− 0.0417	0.9085	0.111	1.291
IV	2.08511	121.65	0.025	1.059	− 0.08511	0.8130	0.087	1.222
V	2.10000	125.9	0.02	1.047	− 0.1000	0.7943	0.078	1.197
VI	2.14000	138.04	0.02	1.047	− 0.1400	0.7245	0.066	1.164
VII	2.17380	149.2	0.015	1.035	− 0.1738	0.6702	0.0556	1.137

Table 33. Logarithmic and Natural Equations of Seven Type Mortality Curves

Type	$Y =$ log k	+ (log s)x	− log $g \times c^x$
I	$Y = 2.0000073$	$+ \quad 0x$	$- 0.000007291\ (2.83)^x$
II	$Y = 2.000204$	$+ \quad 0x$	$- 0.0002042\ (2.018)^x$
III	$Y = 2.04170$	$+ 0.02x$	$- 0.0417\ (1.291)^x$
IV	$Y = 2.08511$	$+ 0.025x$	$- 0.08511\ (1.222)^x$
V	$Y = 2.1000$	$+ 0.02x$	$- 0.1000\ (1.197)^x$
VI	$Y = 2.1400$	$+ 0.02x$	$- 0.1400\ (1.164)^x$
VII	$Y = 2.1738$	$+ 0.015x$	$- 0.1738\ (1.137)^x$
	$y = k$	$\cdot\ s^x$	$\cdot\ g^{c^x}$
I	$y = 100.0$	$\cdot\ 1.00^x$	$\cdot\ 0.9999^{2.83^x}$
II	$y = 100.05$	$\cdot\ 1.00^x$	$\cdot\ 0.9995^{2.018^x}$
III	$y = 110.1$	$\cdot\ 1.047^x$	$\cdot\ 0.9085^{1.291^x}$
IV	$y = 121.65$	$\cdot\ 1.059^x$	$\cdot\ 0.8130^{1.222^x}$
V	$y = 125.9$	$\cdot\ 1.047^x$	$\cdot\ 0.7943^{1.197^x}$
VI	$y = 138.04$	$\cdot\ 1.047^x$	$\cdot\ 0.7245^{1.164^x}$
VII	$y = 149.2$	$\cdot\ 1.035^x$	$\cdot\ 0.6702^{1.137^x}$

in the case of physical property. The formula, therefore, does not have to be restricted to age limits but can be used over the

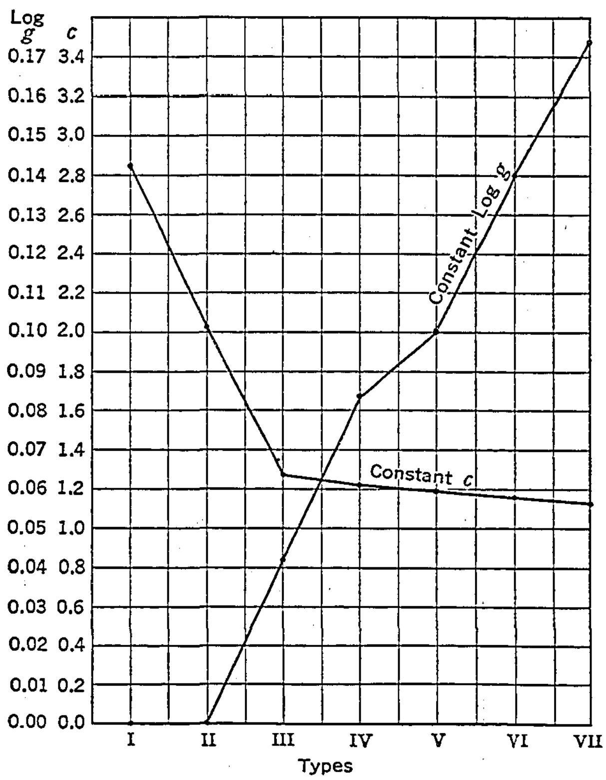


Figure 57. Variation in Constants Logs g and c with Types

entire range from 0 age to maximum life. In fact, one of the practical applications of the formula is the forecasting of useful life from incomplete historical data.

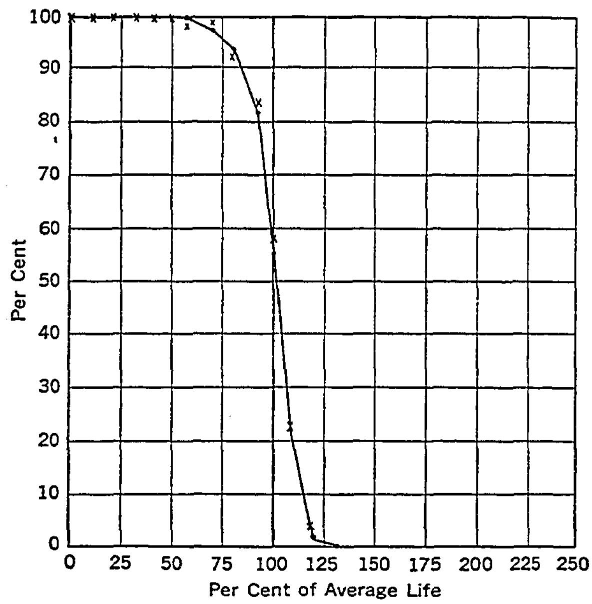


Figure 58. Comparison of Type I Mortality Curve with Values (x) Obtained from Equation: $y = 100.0 \times 1.00^x \times 0.9999^{283^x}$

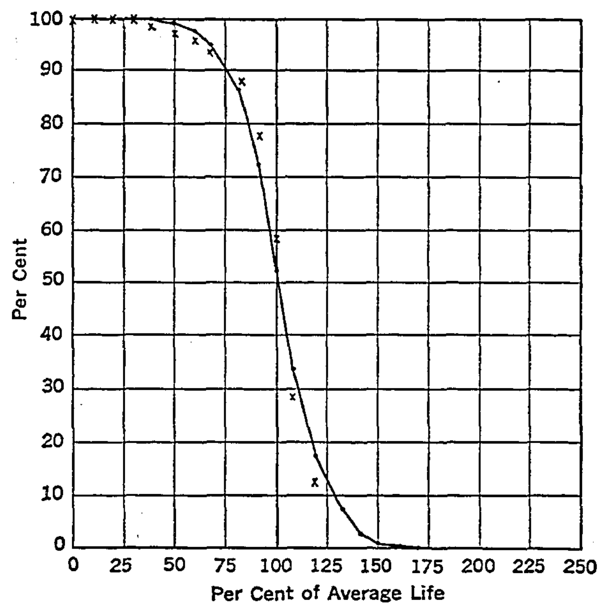


Figure 59. Comparison of Type II Mortality Curve with Values (x) Obtained from Equation: $y = 100.05 \times 1.00^x \times 0.9995^{2.018^x}$

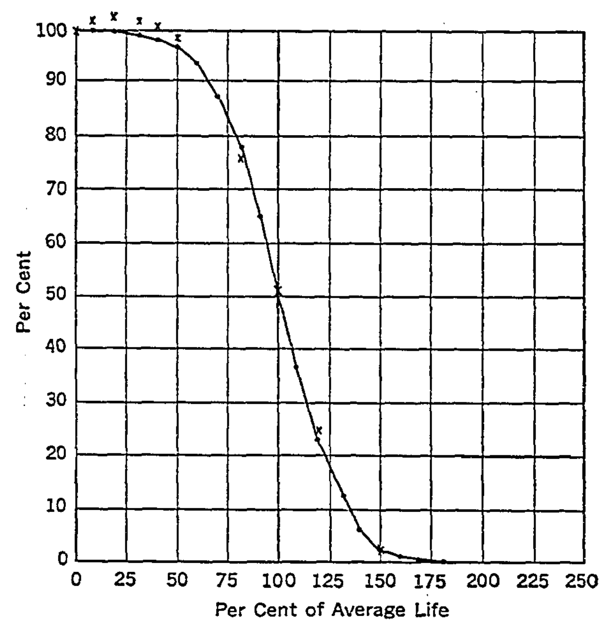


Figure 60. Comparison of Type III Mortality Curve with Values (x) Obtained from Equation: $y = 110.1 \times 1.047^x \times 0.9085^{1.291^x}$

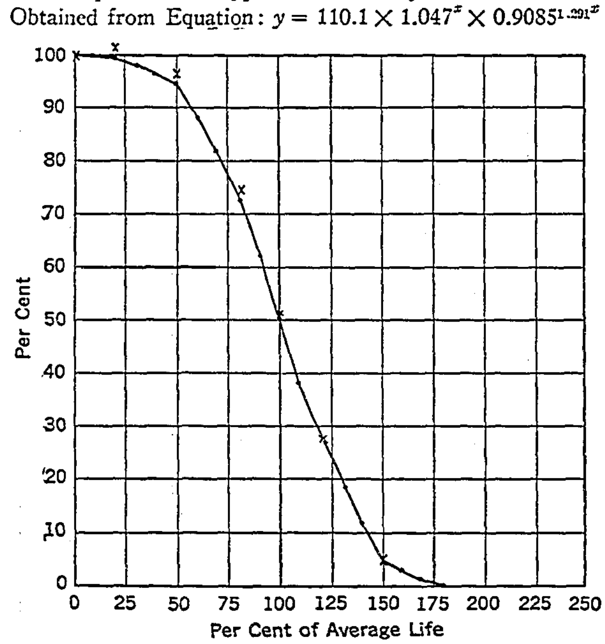


Figure 61. Comparison of Type IV Mortality Curve with Values (x) Obtained from Equation: $y = 121.65 \times 1.059^x \times 0.8130^{1.222^x}$

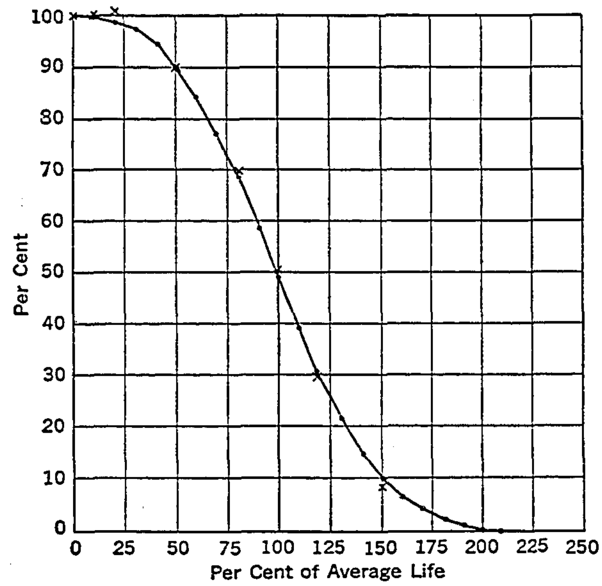


Figure 62. Comparison of Type V Mortality Curve with Values (x) Obtained from Equation: $y = 125.9 \times 1.047^x \times 0.7943^{1.197^x}$

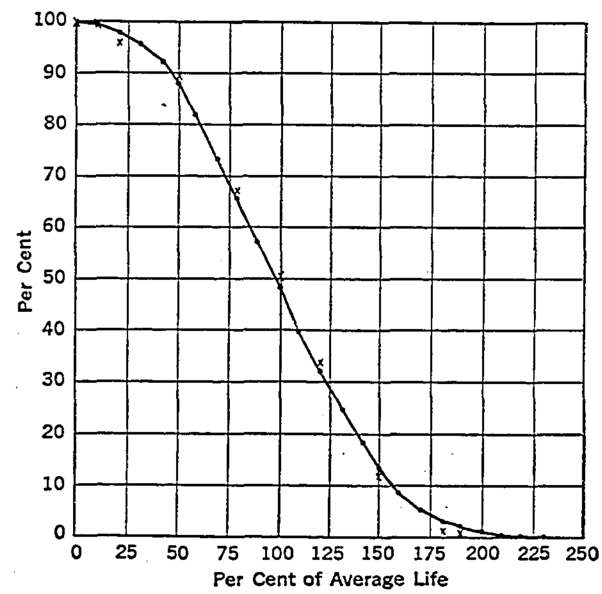


Figure 63. Comparison of Type VI Mortality Curve with Values (x) Obtained from Equation: $y = 138.04 \times 1.047^x \times 0.7245^{1.164^x}$

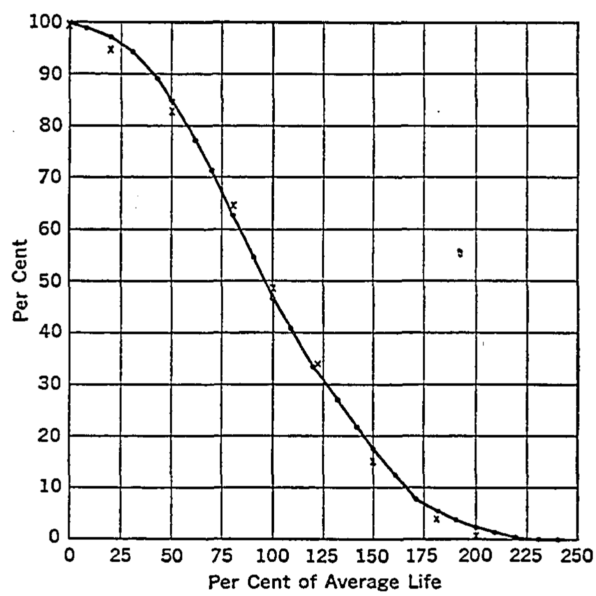


Figure 64. Comparison of Type VII Mortality Curve with Values (x) Obtained from Equation: $y = 149.2 \times 1.035^x \times 0.6702^{1.137^x}$

CHAPTER 6

AVERAGE LIFE

Definition.—The service life of a unit of physical property is the period during which it is capable of rendering efficient service. The period of service begins when the unit is put into operation new and ends at the time the unit is discarded because of renewal or replacement. The service lives of individual units of the same class of property are not the same, as has already been pointed out in previous chapters. Some units of a given group are replaced soon after being put in service, while others remain in service many years. In this respect the life of physical property is quite analogous to the life of human beings. Many humans die in infancy, while others die at all ages between infancy and 100 years of age.

The average life of a unit of property is the average value of a substantial number of individual service lives of the same class of property. To obtain reliable values for average lives of various classes of physical property it thus becomes necessary to tabulate the service lives of a large number of units of the same class and then to compute the average lives from these data. The 52 mortality tables of physical property already referred to are based on such data, and the method of calculating average lives therefrom will be discussed in the following paragraphs.

Method of Computing.—Since the ordinates to the mortality curve shown in Figure 65 represent the survivors to the given ages of the total number of units at age 0, and the abscissae to the same curve represent years in service, it follows that the total area under the curve represents the "unit-service years" of the entire group of units. To obtain the "average life" it is only necessary to divide this area expressed in "unit-years" by the number of units in the group at age 0, thus:

$$\text{Average Life in Years} = \frac{\text{Total Unit-Years}}{\text{Units in Service at Age 0}}$$

In case the survivors are expressed in per cent, the area under the curve will be expressed as "per cent-service years."

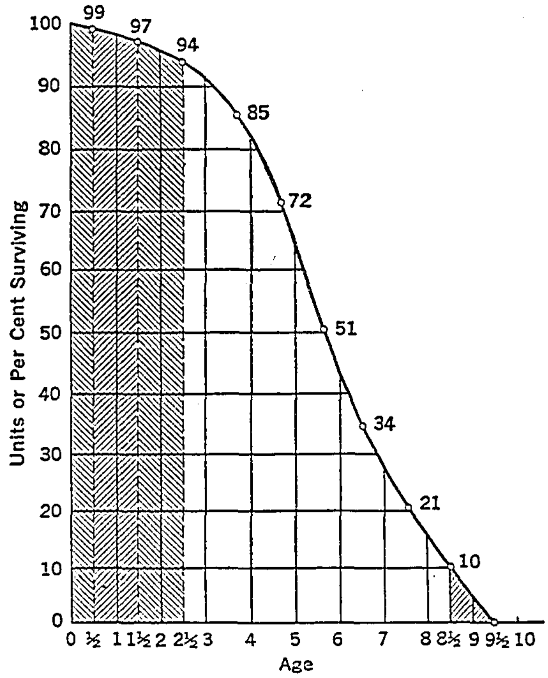


Figure 65. Method of Evaluating Area Under Curve by Dividing Area into Vertical Strips

The average life is obtained by dividing the area in "per cent-years" by the per cent in service at age 0, usually taken as 100%.

If a vertical line is drawn through the age equal to the "average life" thus obtained, the rectangle formed will have the same area as the area under the mortality curve. Such a rectangle is really a mortality curve based on the assumption that all units remain in service until the age equal to the "average life" is reached, when they all go out of service simultaneously. The mortality curve is horizontal out to an age corresponding to "average life" and then decreases abruptly to 0. Such a rectangle, however, does not show how the area under the mortality curve is distributed, which is essential in a study of expectancies. (See Chapter 7.)

The "average life" for each of the 52 groups of physical property was computed in this manner. Figure 66 illustrates the heavy vertical line erected at the age equal to average life.

Plotting Per Cent Survivors.—In the fundamental data the number of units reaching the nearest full year of service when removed are shown. Such units are actually removed from service during the period from ½ year preceding to ½ year following the ages shown. The age interval during which units are removed, therefore, begins ½ year before the age shown. Also, the per cents survivors shown as being in service at a given age, in reality should be shown as being in service at the beginning of the age interval beginning ½ year preceding the given age. All of the observational data in which it was evident that the ages reached in service were recorded to the nearest full year were handled in this manner. Through failure to take this into account some of the mortality tables which have been published elsewhere heretofore, and are republished herewith, were in error by amounts ranging from 0 to ½ year in the values which they show for ages corresponding to given percentages of survivors.

Calculating Area Under Curve from Original Data.—In order to make accurate calculations of the area under the mortality curve it is necessary to divide the area into vertical strips. Figure 65 shows these strips in a typical case. The area of the first strip, ½ year in width, is $\frac{1}{2}\left(\frac{100+99}{2}\right) = 49.75\%$-years.

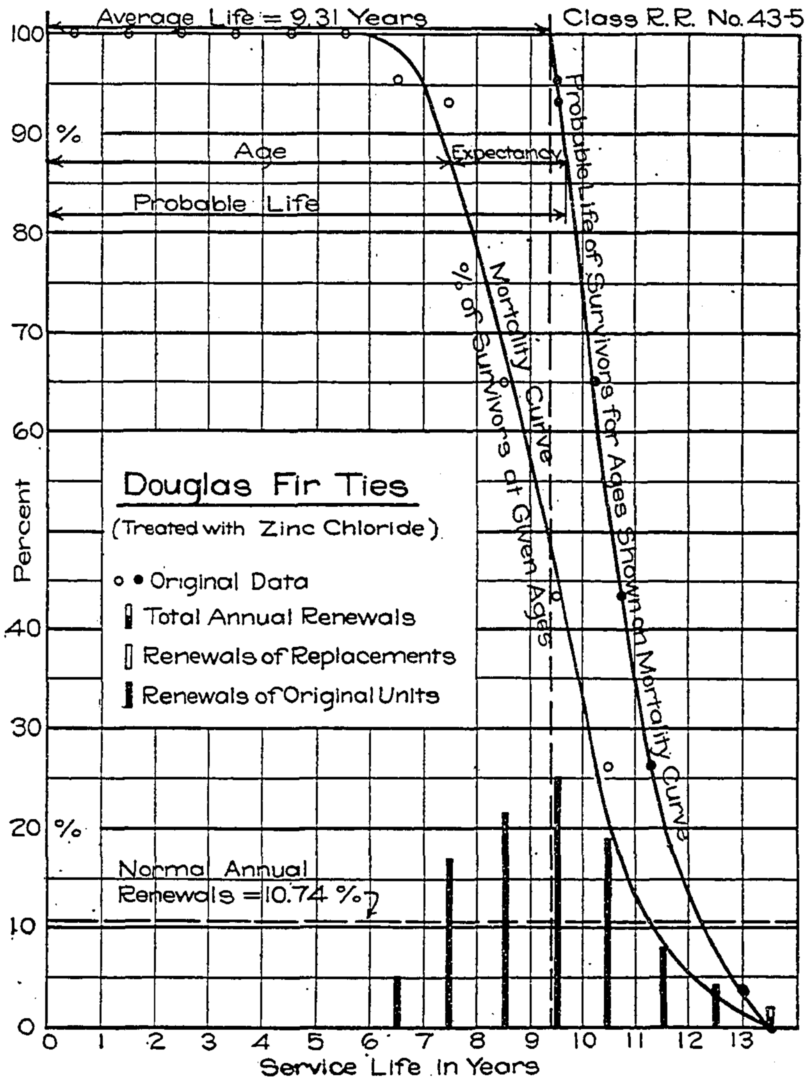


Figure 66. Method of Indicating Average Life on Diagram of Life Characteristics

The next strip is 1 year wide and its average ordinate is $\frac{99 + 97}{2} = 98$. Therefore, its area $= 98 \times 1 = 98$%-years. The area of the third strip in like manner is $1 \left(\frac{97 + 94}{2}\right) =$ 95.5%-years. This process is continued until the areas of all the strips have been evaluated. The average ordinate of the last strip is obtained by averaging $\frac{10 + 0}{2} = 5$ and its area $= 5 \times 1 = 5$%-years. The sum of all the strip areas is then obtained and this area divided by 100% gives the average life of the individual units of the group.

Short-Cut Area Calculation.—It will be noted that in the above calculations of areas from original data plotted on half-year points, all ordinates are taken twice and divided by two except the first two ordinates at 0, and at ½ year values. Since all the remaining ordinates are taken twice and also divided by two, the two operations cancel each other and leave the ordinate as the value of the area. Only ¼ of the first ordinate is taken as the 100 was divided by 4. The second ordinate enters into two calculations, once when ¼ of it is taken and again when ½ of it is taken. The sum of these operations on the second ordinate is equivalent to taking $(\frac{1}{4} + \frac{1}{2}) = \frac{3}{4}$ of its value.

The short-cut rule for total area under the mortality curve, therefore, consists in taking ¼ of ordinate at 0 age, ¾ of ordinate at ½ year age, and the sum of all other ordinates on the half-year points. This gives the same area as is obtained by summing up the areas of the various strips.

Relation of Average Life to Median and to Mode.—As the median and the mode are important statistical constants of any frequency series, the relation of the average life to these constants is of interest. The median year and modal year were read directly from the mortality charts Nos. 1-52, the median

Table 34. Average Life, Median and Modal Years of 52 Mortality Groups, and Ratios of Average Life to Median and Modal Year

Mortality Chart Number	Average Life	Median Year	Modal Year	Ratio of	
				Average Life to Median Year	Average Life to Modal Year
(1)	(2)	(3)	(4)	(5)	(6)
1............	17.00	15.60	10.0	1.09	1.70
2............	13.48	12.00	10.0	1.12	1.35
3............	21.32	20.00	16.0	1.07	1.33
4............	31.47	30.00	27.0	1.05	1.16
5............	14.81	15.00	16.0	0.99	0.93
6............	5.27	5.30	7.0	1.00	0.75
7............	11.57	12.30	13.0	0.96	0.89
8............	12.53	12.00	8.0	1.04	1.57
9............	8.72	7.00	6.0	1.25	1.45
10............	10.63	9.60	7.0	1.11	1.52
11............	9.68	9.50	10.0	1.02	0.97
12............	9.26	9.60	13.0	0.97	0.71
13............	11.40	11.80	14.0	0.97	0.82
14............	14.20	14.20	16.0	1.00	0.89
15............	14.87	13.80	14.0	1.08	1.06
16............	16.25	16.25	16.0	1.00	1.01
17............	12.26	12.60	15.0	0.97	0.82
18............	12.26	11.00	8.0	1.11	1.53
19............	11.62	9.00	7.0	1.29	1.66
20............	8.33	7.40	7.0	1.13	1.19
21............	9.84	9.50	10.0	1.04	0.98
22............	9.67	9.10	5.0	1.06	1.94
23............	9.20	8.60	7.0	1.07	1.31
24............	10.68	10.68	11.0	1.00	0.97
25............	9.93	9.40	9.0	1.06	1.10
26............	11.47	10.70	10.0	1.07	1.15
27............	7.98	7.70	7.0	1.04	1.14
28............	10.00	10.00	10.0	1.00	1.00
29............	12.30	12.70	14.0	0.97	0.88
30............	10.52	11.00	13.0	0.96	0.81
31............	30.35	30.00	30.0	1.01	1.01
32............	23.58	22.00	17.0	1.07	1.39
33............	25.39	24.50	21.0	1.04	1.21
34............	33.15	33.00	30.0	1.01	1.10
35............	19.97	20.70	21.0	0.97	0.95
36............	20.81	21.60	21.0	0.96	0.99
37............	18.66	19.60	21.0	0.95	0.89
38............	19.31	20.00	22.0	0.97	0.88
39............	11.18	11.60	13.0	0.97	0.86
40............	9.03	9.00	11.0	1.00	0.82
41............	13.36	13.40	15.0	1.00	0.89
42............	17.19	17.60	19.0	0.98	0.90
43............	9.31	9.30	10.0	1.00	0.93
44............	10.96	11.40	12.0	0.96	0.91
45............	10.47	10.60	12.0	0.99	0.87
46............	11.23	11.40	13.0	0.98	0.87
47............	10.19	10.20	11.0	1.00	0.93
48............	9.78	9.60	10.0	1.02	0.98
49............	11.44	11.60	14.0	0.99	0.82
50............	10.79	9.70	10.0	1.11	1.08
51............	8.30	8.30	9.0	1.00	0.92
52............	8.17	8.30	9.0	0.99	0.91
			Totals =	53.46	55.70
			Average Ratio =	$\frac{53.46}{52}$	$\frac{55.70}{52}$
			=	1.028	1.071

year being the year at which 50% of the units survive and the modal year being the year during which the largest number or per cent of the units of the group go out of service. In Table 34 the average life and the median and modal years are

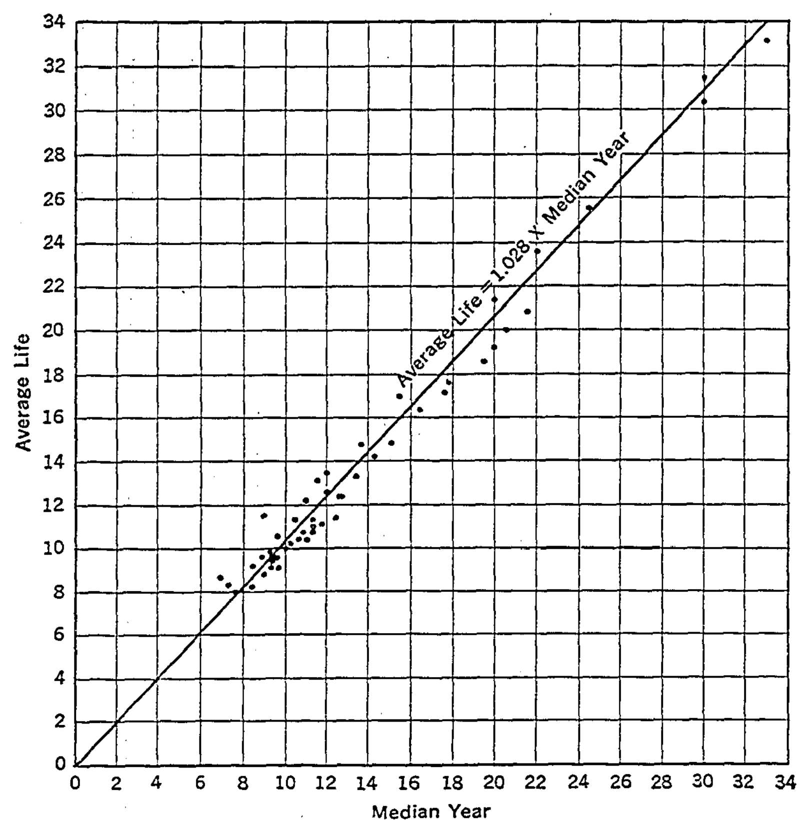


Figure 67. Relation Between Average Life and Median Year for 52 Property Groups

tabulated for each of the 52 property groups. The ratios of the average life to the median year and to the modal year have also been calculated and are tabulated in columns (5) and (6) of the table.

If the average life values given in the table are plotted against the corresponding values for median and modal years the diagrams shown in Figures 67 and 68 result. In Figure 67 which shows the relation between average life and median year, the points indicate a pronounced relationship. The degree

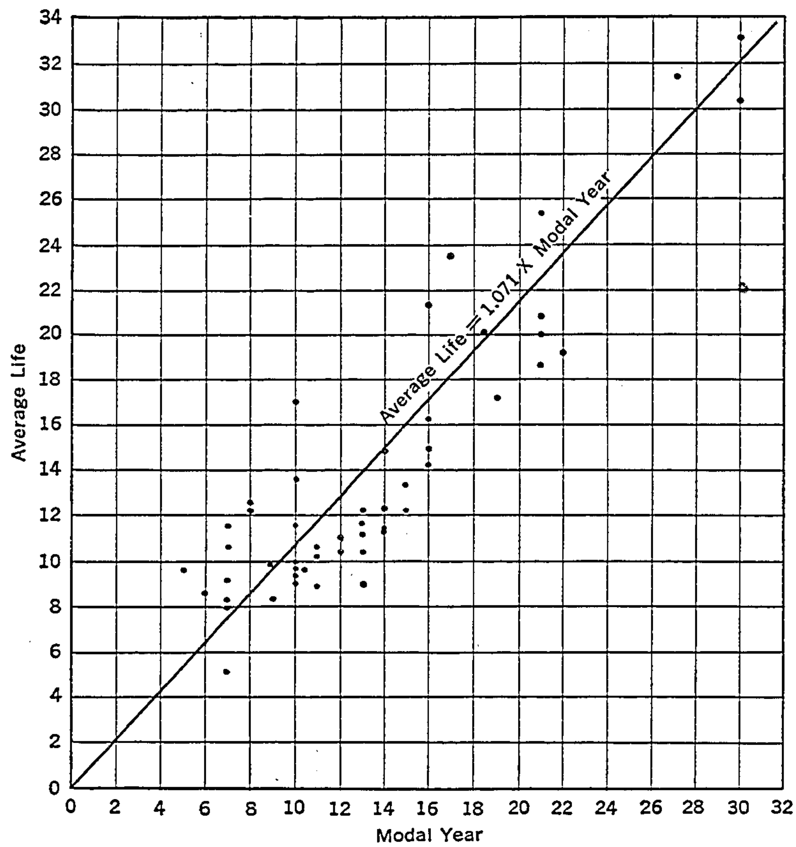


Figure 68. Relation Between Average Life and Modal Year for 52 Property Groups

to which the ratios vary from the average is shown in Tables 35 and 36. In these tables the number of cases out of 52, in which the ratios lie in each interval are shown. The fact that the average life and median year are closely related is shown by the fact that over half of the cases, 27 in number, lie in one interval from 0.96–1.00. Similar ratios of average life to mode show a much greater spread. The greatest variations from the average are in the upper range. The range above is to 2, whereas below the range is only to 0.70. This accounts for the fact that although more than half of the cases are below 1.00 the average of all cases is greater than 1.00.

An average value for these two linear relations can be obtained by taking the average of the respective ratios. By referring to Table 34 the total of the 52 ratios of average life to median year is 53.46, making the average ratio equal to $\frac{53.46}{52} = 1.028$. Likewise, the sum of the ratios of average life to mode is 55.70, making the average ratio equal to $\frac{55.7}{52} = 1.071$.

The average life thus exceeds the median and mode by 2.8% and 7.1% respectively.

Table 35. Distribution of Cases of Ratio of Average Life to Median Year by Intervals

Interval	Number of Cases of Ratio of *Average Life* to Median Year in Interval
0.91–0.95	1
0.96–1.00	27
1.01–1.05	9
1.06–1.10	8
1.11–1.15	5
1.16–1.20	0
1.21–1.25	1
1.26–1.30	1
	52

Table 36. Distribution of Cases of Ratio of Average Life to Modal Year by Intervals

Interval	Number of Cases of Ratio of *Average Life* to Modal Year in Interval
0.70–0.80	2
0.81–0.90	15
0.91–1.00	13
1.01–1.10	6
1.11–1.20	4
1.21–1.30	1
1.31–1.40	4
1.41–1.50	1
1.51–1.60	3
1.61–1.70	2
1.71–1.80	0
1.81–1.90	0
1.91–2.00	1
	52

If a straight line be drawn on Figure 67 such that the relation

$$\text{Average Life} = 1.028 \text{ Median Year}$$

is observed, the line will show the relation between any value of average life and median year or vice versa. This line represents the relationship sought. It will be noted that the line goes through the origin, that is, there is no intercept on the ordinate.

In a like manner if a straight line be drawn on Figure 68 such that the relation

$$\text{Average Life} = 1.071 \text{ Modal Year}$$

is observed, the line will show the relation between any value of average life and mode or vice versa. This line also goes through the origin.

The deduction that the average life is almost equal to the median means that the area *A* is approximately equal to area *B*. (See Figure 69.) When the average life line is drawn on the mortality chart it forms a rectangle the area of which is equal to the area under the mortality curve. The rectangle formed by the median will enclose the same area unless *A* and *B* differ in area. Therefore, when *A* and *B* are equal the median and average life line coincide. If *A* is much larger than *B* the average life will be considerably greater than the median, and if *A* is smaller than *B* the average life is smaller than the median.

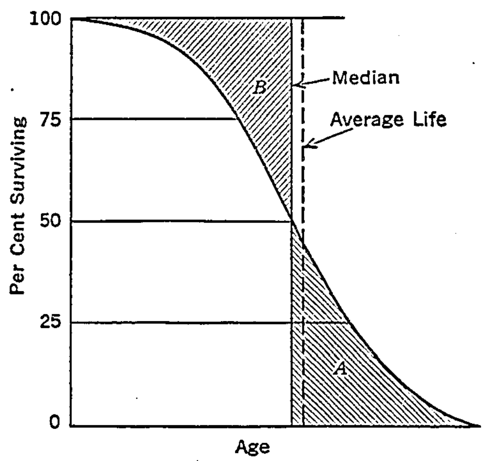


Figure 69. Relation of Average Life and Median

Relation of Average Life, Median Year, and Modal Year to Maximum Life.—The determination of the relation of the average life, median year, and modal year to the maximum life of a group involves an analysis of the ratios of average life to maximum life, median year to maximum life, and modal year to maximum life. Table 37 gives the values of average life, median, and mode, as well as the ratios just enumerated for the 52 groups of physical property.

The distribution of these ratios is shown in Tables 38, 39, and 40. The average life seems to have a somewhat definite relation to the maximum life as the distribution curve of the

Table 37. Average Life, Median Year, Modal Year, Maximum Life, and Ratios of Average Life to Median Year, Modal Year, and Maximum Life

Mortality Chart Number	Average Life	Median Year	Modal Year	Maximum Life	Ratio of		
					Average Life to Max. Life	Median Year to Max. Life	Modal Year to Max. Life
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	17.00	15.60	10.0	34.5	0.49	0.45	0.29
2	13.48	12.00	10.0	34.5	.39	.35	.29
3	21.32	20.00	16.0	50.5	.42	.40	.32
4	31.47	30.00	27.0	51.5	.61	.58	.52
5	11.81	15.00	16.0	23.5	.63	.64	.68
6	5.27	5.30	7.0	13.5	.39	.39	.52
7	11.57	12.30	13.0	15.5	.74	.79	.84
8	12.53	12.00	8.0	26.5	.47	.45	.30
9	8.72	7.00	6.0	25.5	.34	.27	.24
10	10.63	9.60	7.0	24.5	.43	.39	.29
11	9.68	9.50	10.0	19.5	.50	.49	.51
12	9.26	9.60	13.0	17.5	.53	.55	.74
13	11.40	11.80	14.0	21.5	.53	.55	.65
14	14.20	14.20	16.0	27.5	.52	.52	.58
15	14.87	13.80	14.0	38.5	.39	.36	.36
16	16.25	16.25	16.0	28.5	.57	.57	.56
17	12.26	12.60	15.0	23.5	.52	.54	.64
18	12.26	11.00	8.0	26.5	.46	.41	.30
19	11.62	9.00	7.0	28.5	.41	.32	.25
20	8.33	7.40	7.0	19.5	.43	.38	.36
21	9.84	9.50	10.0	23.5	.42	.40	.43
22	9.67	9.10	5.0	24.5	.40	.37	.20
23	9.20	8.60	7.0	21.5	.43	.40	.33
24	10.68	10.68	11.0	23.5	.45	.45	.47
25	9.93	9.40	9.0	24.5	.41	.38	.37
26	11.47	10.70	10.0	22.5	.51	.48	.44
27	7.98	7.70	7.0	14.0	.57	.55	.50
28	10.00	10.00	10.0	20.0	.50	.50	.50
29	12.30	12.70	14.0	17.5	.70	.73	.80
30	10.52	11.00	13.0	18.0	.58	.61	.72
31	30.35	30.00	30.0	85.0	.36	.35	.35
32	23.58	22.00	17.0	47.5	.50	.46	.36
33	25.39	24.50	21.0	50.5	.50	.49	.47
34	33.15	33.00	30.0	54.5	.61	.61	.55
35	19.97	20.70	21.0	37.5	.53	.55	.56
36	20.81	21.60	21.0	30.5	.68	.71	.69
37	18.66	19.60	21.0	32.5	.57	.60	.65
38	19.31	20.00	22.0	33.5	.58	.60	.66
39	11.18	11.60	13.0	15.5	.72	.75	.84
40	9.03	9.00	11.0	15.5	.58	.58	.71
41	13.36	13.40	15.0	15.5	.86	.87	.97
42	17.19	17.60	19.0	19.5	.88	.90	.98
43	9.31	9.30	10.0	13.5	.69	.69	.74
44	10.96	11.40	12.0	15.5	.71	.73	.77
45	10.47	10.60	12.0	14.5	.72	.73	.83
46	11.23	11.40	13.0	14.5	.77	.79	.90
47	10.19	10.20	11.0	14.5	.70	.70	.76
48	9.78	9.60	10.0	17.5	.56	.55	.57
49	11.44	11.60	14.0	14.5	.79	.80	.97
50	10.79	9.70	10.0	19.5	.55	.50	.51
51	8.30	8.30	9.0	13.5	.61	.61	.67
52	8.17	8.30	9.0	10.5	.78	.79	.86
				Totals =	28.99	28.63	29.32
				Average Ratio =	28.99/52	28.63/52	29.32/52
				=	0.5575	0.551	0.564

Table 38. Distribution of Cases of Ratio of Average Life to Maximum Life by Intervals

Interval	Number of Cases of Ratio of *Average Life* to Maximum Life in Interval
0.30–0.35	1
.36– .40	5
.41– .45	8
.46– .50	7
.51– .55	7
.56– .60	7
.61– .65	4
.66– .70	4
.71– .75	4
.76– .80	3
.81– .85	0
.86– .90	2
	52

Table 39. Distribution of Cases of Ratio of Median Year to Maximum Life by Intervals

Interval	Number of Cases of Ratio of *Median Year* to Maximum Life in Interval
0.26–0.30	1
.31– .35	3
.36– .40	9
.41– .45	4
.46– .50	6
.51– .55	7
.56– .60	5
.61– .65	4
.66– .70	2
.71– .75	5
.76– .80	4
.81– .85	0
.86– .90	2
	52

Table 40. Distribution of Cases of Ratio of Modal Year to Maximum Life by Intervals

Interval	Number of Cases of Ratio of *Modal Year* to Maximum Life in Interval
0.21–0.25	3
.26– .30	5
.31– .35	3
.36– .40	4
.41– .45	3
.46– .50	3
.51– .55	5
.56– .60	4
.61– .65	3
.66– .70	4
.71– .75	4
.76– .80	3
.81– .85	3
.86– .90	2
.91– .95	0
.96–1.00	3
	52

ratios is of the usual form. The median year is more variable, and the mode occurs at any place over the range with almost the same frequency.

In Table 37 are also given the totals for the 52 ratios and the average ratio. The values for the averages of the 52 ratios are as follows:

$$\text{Ratio of } \frac{\text{Average Life}}{\text{Maximum Life}} = 0.5575$$

$$\text{Ratio of } \frac{\text{Median Year}}{\text{Maximum Life}} = 0.551$$

$$\text{Ratio of } \frac{\text{Modal Year}}{\text{Maximum Life}} = 0.564$$

These averages would indicate that the mortality curves are in general unsymmetrical about a vertical axis drawn through either average life, median year, or modal year, since the vertical axis is displaced to the right about 5% to 7% of the central position. Mortality curves are, therefore, not true probability curves as the units do not have an equal chance of being replaced early as well as late in life, but have a better chance of staying in service during the early years. Instead of a heavy infant mortality there is apparently a reduced hazard. This agrees with experience as units which become inadequate early in life can oftentimes be moved to another location where they are adequate and where they can live out the remainder of their useful lives.

Table 41. Per Cent Surviving Average Life of 52 Mortality Groups

Mortality Chart Number	Per Cent Survivors at Average Life	Mortality Chart Number	Per Cent Survivors at Average Life
1	45	29	56
2	42	30	54
3	44	31	48
4	42	32	45
5	50	33	46
6	51	34	48
7	63	35	57
8	48	36	58
9	42	37	58
10	43	38	55
11	48	39	60
12	53	40	50
13	52	41	50
14	51	42	58
15	43	43	49
16	50	44	57
17	52	45	50
18	43	46	52
19	38	47	48
20	37	48	46
21	48	49	52
22	46	50	34
23	44	51	49
24	50	52	50
25	45		
26	44		
27	46	Total =	2,540
28	50	Average =	48.85%

Table 42. Distribution of Cases of Per Cent Survivors by Intervals

Interval	Number of Cases of Per Cent Survivors in Interval
0.31–0.35	1
.36– .40	2
.41– .45	12
.46– .50	20
.51– .55	9
.56– .60	7
.61– .65	1
	52

Per Cent Survivors at Average Life.—It is of interest to observe the per cent of the original units of the group which survive the average life of the group. In the case of a perfectly symmetrical curve, the per cent survivors would be 50. Table 41 shows the per cent survivors at average life for the 52 mortality groups. These values were read directly from the mortality charts and are the per cent survivors where the average life line crosses the survivor curve. The average of these percentages is 48.85 which is close to the value a normal frequency group would have.

The distribution of these percentages is shown in Table 42. Twelve cases have percentages lying between 41% and 45%, 20 between 46% and 50%, and 9 between 51% and 55%. The total number of cases below 50% is 35, and the number of cases above 50% is 17. The frequency curve is very peaked indicating that the spread is not wide.

CHAPTER 7

EXPECTANCY AND PROBABLE LIFE

Definition of Life Expectancy.—Just as "average life" is the anticipated future life at age 0 of an average unit of property, so "life expectancy" is the anticipated future life at any given age of an average unit. From this it follows that the life expectancy at age 0 becomes the average life. Average life is, therefore, only a special case of life expectancy. Moreover, just as the area under the mortality curve represents the entire unit-service years, or per cent-service years available from the group of units at age 0, so the area under the mortality curve to the right of any age ordinate represents the remaining unit-service years available from the units surviving that age. Hence, the expectancy at any given age is equal to the quotient of the total area under the mortality curve to the right of the ordinate at the given age, divided by the number of units surviving the given age.

Method of Computing Life Expectancy from Original Data.—To determine the life expectancy at a given age it is, thus, only necessary to evaluate the area under the mortality curve to the right of the given age ordinate and to divide this area by the survivors at that age. If the ordinate is expressed in number of units of property the area will be expressed in unit-years, and the life expectancy is obtained by dividing this area by the number of units still in service at that age. If the ordinates are given in per cent of the number of units in the group at age 0, the area will be expressed in per cent years, and the life expectancy is obtained by dividing this area by the per cent units in service at that age.

The areas corresponding to the various ages can best be evaluated by dividing the area under the curve into vertical strips 1 year wide, as outlined in Chapter 6. By subtotaling the areas of these strips from right to left, the area to the right of each age ordinate will be made available. Table 43 illustrates the method of subtotaling these strip areas from right to left for the mortality curve shown in Figure 65 in Chapter 6. The necessity for plotting the per cent survivors of the original observational data on the half-year values was also discussed in the preceding chapter. Starting at the bottom of the table which corresponds to the right end of the mortality curve, the area of the first strip between 8½ and 9½ years is equal to $\frac{10.0 + 0}{2} = 5$ per cent-years. The area of the next strip between 7½ and 8½ years is $\frac{21 + 10}{2} = 15.50$ per cent-years, etc., up to the extreme left strip between 0 and ½ year which is only ½ year wide. The area of this strip is $\frac{100 + 99}{4} =$ 49.75 per cent-years. The values for all the strips are shown in column (4) of the table, and the cumulative subtotals of these strip areas are given in column (5). The life expectancies for the various ages are given in column (6) and were obtained by dividing the area in the subtotal area column by the corresponding survivors. The expectancies were calculated in this manner for the 52 mortality curves listed in Chapter 2.

Definition of Probable Life.—The probable life at any given age is here defined as the length of life that a unit of property is expected to live at that age. It is the sum of the already expired life at the given age plus the unexpired life or expectancy at that age. Since the already expired life at a given age is the age itself, the expression for probable life becomes:

$$\text{Probable Life at a Given Age} = \text{Age} + \text{Expectancy at That Age}$$

Table 43. Method of Computing Life Expectancies from Original Data by Use of Strip Areas

Age in Years (1)	Renewals of Original Units During Year Following Given Ages—Per Cent (2)	Survivors at Given Ages—Per Cent (3)	Partial Areas Under Curve Service-Per Cent Years (4)	Remaining Service "Per Cent-Years" (5)	Expectancies in Years (6)	Probable Life in Years (7)
0	1	100	49.75	563.25	5.63	5.63
½	2	99	98.00	513.50	5.19	6.19
1½	3	97	95.50	415.50	4.28	5.78
2½	9	94	89.50	320.00	3.41	5.91
3½	13	85	78.50	230.50	2.77	6.27
4½	21	72	61.50	152.00	2.11	6.61
5½	17	51	42.50	90.50	1.77	7.27
6½	13	34	27.50	48.00	1.41	7.91
7½	11	21	15.50	20.50	0.98	8.48
8½	10	10	5.00	5.00	0.50	9.00
9½		0				

Computing and Plotting Probable Life.—The expression just given for probable life indicates the simple manner of computing the probable life at any age if the expectancy at that age is known. (See Table 43.) The expectancies were computed for the 52 property groups, and the probable lives calculated therefrom were plotted on the mortality charts in the following manner (see Figure 70): At age 0 the expectancy is the average life and is represented on the chart by a vertical line drawn through the age equal to the average life. At age 0 the average life is also equal to the probable life because the expired life is still 0; in fact, at age 0,

$$\text{Expectancy} = \text{Average Life} = \text{Probable Life}$$

For any other age, the expectancy is shown by means of a horizontal line drawn from the intersection of the age line and the mortality curve and equal in length to the expectancy measured on the age scale. The combined length of age and expectancy is the probable life. The extreme right end of the expectancy line, therefore, is a point on the probable life curve. This method of plotting makes possible reading age, expectancy, and probable life on the horizontal age scale. Figure 71 shows these quantities in place on one of the 52 charts. On all of the mortality charts the probable lives calculated from

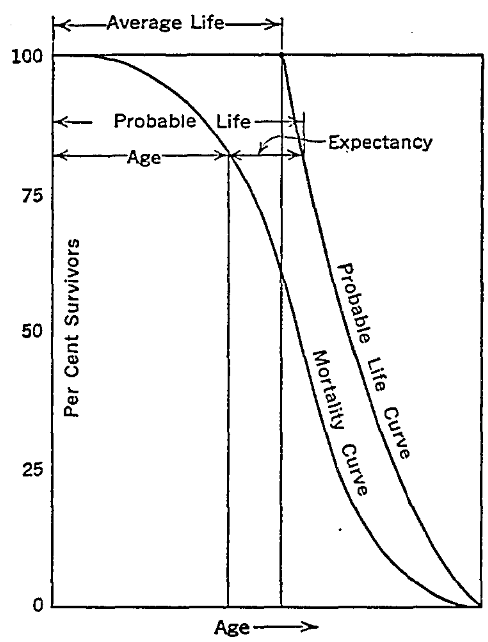


Figure 70. Method of Plotting Expectancy and Probable Life on Mortality Charts

the original observational data were plotted on the half-years and are shown by circles. The curves drawn through these points do not in all cases fit exactly throughout their length. This is in part due to a smoothing process that was applied to the survivor curve. This process is discussed in the following paragraph. Therefore, to obtain the total probable life of average survivors at any given age, find the age at the bottom of the diagram, trace vertically upward to the mortality curve

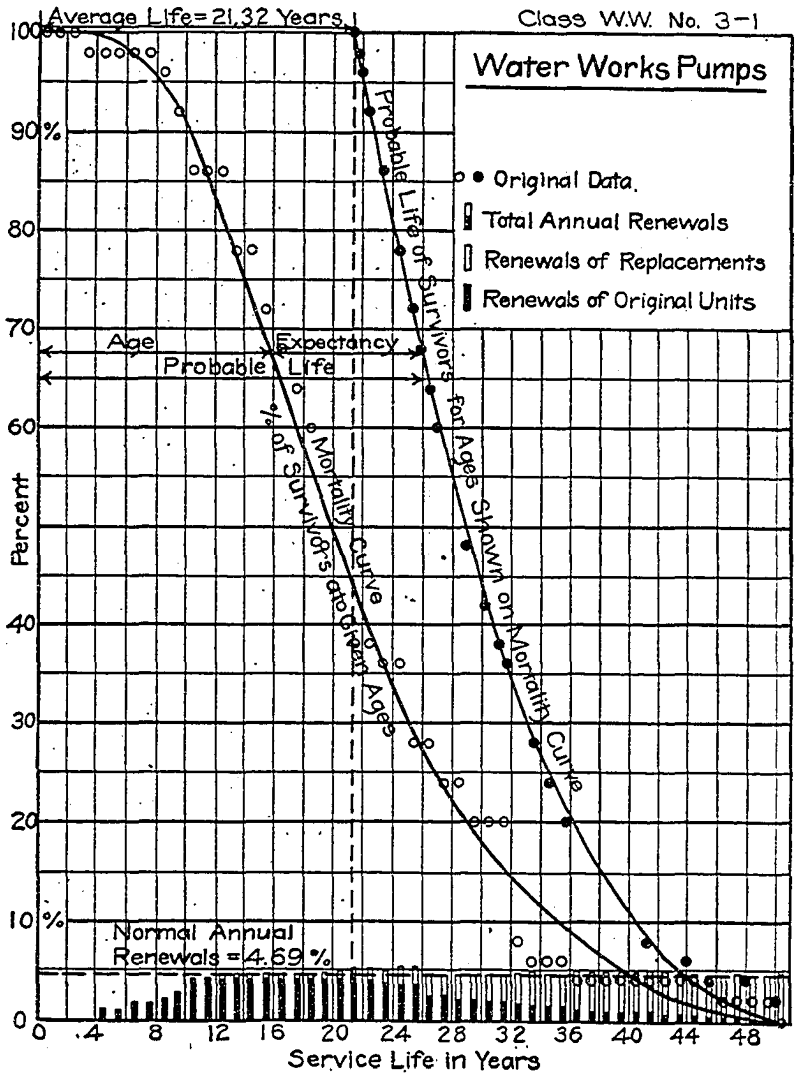


Figure 71. Mortality Chart of Waterworks Pumps Showing Method of Plotting Expectancy and Probable Life

and thence horizontally to the right to the probable life curve; then drop to the age scale and read the probable life at the bottom of the diagram.

Adjusting the Mortality Data and Life Expectancies.—At present, in most cases, the life experience data of physical property are not yet in sufficient mass to insure smooth mortality curves like those of humans shown in Figure 3. As a result, the curves are slightly irregular in places. Such curves will not give the true expectancies and probable lives as reliably as smooth curves drawn through the same points.

Table 44. Life Experience Data of Waterworks Pumps to Accompany Figure 71

(Method of presentation is similar to that used in human life tables.)

Age Interval	Renewals of Original Units in Age Interval	Survivors at Beginning of Age Interval	Expectancy of Survivors at Beginning of Age Interval	Probable Life of Survivors at Beginning of Age Interval
Years	Per Cent	Per Cent	Years	Years
0– 1...........	0.0	100.00	21.32	21.32
1– 2...........	0.0	100.00	20.32	21.32
2– 3...........	0.0	100.00	19.32	21.32
3– 4...........	0.0	100.00	18.32	21.32
4– 5...........	0.8	100.00	17.32	21.32
5– 6...........	0.8	99.2	16.45	21.45
6– 7...........	1.5	98.4	15.58	21.58
7– 8...........	1.5	96.9	14.82	21.82
8– 9...........	2.0	95.4	14.04	22.04
9–10...........	2.5	93.4	13.33	22.33
40–41...........	1.0	4.9	3.58	43.58
41–42...........	1.0	3.9	3.37	44.37
42–43...........	0.5	2.9	3.35	45.35
43–44...........	0.5	2.4	2.95	45.95
44–45...........	0.5	1.9	2.59	46.59
45–46...........	0.4	1.4	2.34	47.34
46–47...........	0.3	1.0	2.08	48.08
47–48...........	0.2	0.7	1.75	48.75
48–49...........	0.2	0.5	1.25	49.25
49–50...........	0.2	0.3	0.75	49.75
50 –50½.......	0.1	0.1	0.25	50.25
50½–51.........	...	0.0	0.00	50.50

Hence, the preparation of actual mortality curves and expectancy and probable life curves and tables requires two sets of calculations, one from the actual original survivor data as already outlined, with no irregularities smoothed out, and the other from a smooth mortality curve fitting the same data as nearly as practicable. As already noted, the calculations from the original observational data are shown on the mortality charts by circles. The results of the second calculation are given in the mortality tables accompanying the charts, and the curves representing these data are drawn on the charts. Examination of the curves will thus show at a glance what liberties were taken in drawing the smooth survivor curves and the extent of the divergence between the final probable life curve and the points shown as circles from the first calculation.

In fitting a smooth mortality curve to the original survivor data great care was taken to fit the original points as nearly as practicable, and to retain exactly the same total area under the curve. The latter feature was essential in order that the average life of the group of units remain unchanged.

The per cent survivors to the smooth mortality curve given in the table accompanying each chart were read on the full years instead of the half years as were the original data. This was done to make the age intervals 0-1, 1-2, 2-3, etc., which is a much more satisfactory interval in which to indicate replacements. Such intervals also facilitate the calculation of annual renewals. (See Chapter 8.) Furthermore, this change makes the manner of presenting mortality statistics of physical property analogous to that used in human mortality tables. An example of such table for Figure 71 is shown herewith as Table 44.

Short-Cut Calculation of Area with Full Year Intervals.—In the process of making the total area under the smooth curve the same as under the original curve, many trials are usually necessary. This requires a frequent totalizing of the strip areas. A short-cut formula for this purpose consists in taking,

> 1/2 of the first ordinate, that is, ordinate at age 0, plus all other ordinates except the last, plus 3/4 of the last ordinate.

The above follows directly from the manner of evaluating the strip areas. In computing the average ordinate of each strip, all ordinates enter in calculation twice except the first ordinate. One-half of the ordinate is taken each time, making the two operations the equivalent of taking the whole ordinate. The ordinate at age 0, however, only enters into the calculations once and therefore only ½ of its value is taken, which is always 50% when the ordinate is expressed in per cent. In the case of the last calculation the strip is only ½ year wide and therefore, instead of dividing by 2 the divisor is 4. This results in taking $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ of the last ordinate.

Types of Life Expectancy Curves.—From the seven graduated type mortality curves discussed in Chapter 4, seven type life expectancy curves result. Instead of calculating the expectancies directly from the seven ungraduated type mortality curves, they were calculated from the seven mortality curves obtained from the frequency equations. It was believed that these results might be more representative of the natural data than would the expectancies calculated from the actual curves assumed. The fact that no serious variations resulted is pointed out in a subsequent paragraph.

The data for the seven type expectancy curves are summarized in Table 45 and the curves plotted therefrom are shown in Figure 72. These curves show at a glance that life expectancy is a decreasing function with age. In Type I the decrease in expectancy is equal to acquired age until shortly before average life is reached. If the first portion or straight line portion of the curve were continued it would almost reach

Table 45. Expectancies in Per Cent of Average Life for Seven Graduated Type Mortality Curves

Age in Per Cent of Average Life	Types						
	I	II	III	IV	V	VI	VII
0..........	99.9	101.5	100.1	99.9	99.8	100.0	99.9
10..........	89.9	91.5	90.1	89.9	90.0	90.3	90.4
20..........	79.9	81.5	80.1	80.1	80.7	81.2	82.1
30..........	69.9	71.5	70.3	70.8	71.9	72.9	74.8
40..........	59.9	61.5	60.7	62.0	63.9	65.4	68.5
50..........	50.0	51.7	51.7	53.9	56.7	58.8	62.7
60..........	40.1	42.4	43.6	46.7	50.3	52.9	57.6
70..........	30.6	34.0	36.4	40.3	44.5	47.7	52.9
80..........	21.8	26.9	30.3	34.8	39.4	43.1	48.5
90..........	14.4	21.1	25.2	30.0	34.8	38.9	44.5
100..........	8.9	16.7	21.0	25.9	30.7	35.1	40.7
110..........	5.3	13.3	17.4	22.2	27.1	31.7	37.1
120..........	5.0	10.6	14.5	19.0	23.8	28.5	33.7
130..........		8.7	12.0	16.1	20.7	25.5	30.5
140..........		7.6	10.0	13.5	17.9	22.8	27.4
150..........		7.0	8.2	10.9	15.4	20.2	24.4
160..........		5.0	6.6	8.1	13.0	17.8	21.5
170..........			5.0	5.0	10.9	15.5	18.7
180..........					8.8	13.4	16.0
190..........					6.5	11.4	13.3
200..........					5.0	9.4	10.8
210..........						7.5	8.2
220..........						5.0	5.9
230..........							5.0

Note: The expectancies at age 0 did not exactly equal 100%, due to the fact that the mortality curves resulting from the frequency equations did not necessarily contain the proper area.

zero expectancy at average life. All of the curves follow this straight line portion of the curve for a distance and then diverge from it. In fact, the straight line from 100% at 0 age to 0 expectancy at 100% age is sort of an asymptote which all of the curves attempt to approach. This straight line is the expectancy curve for a mortality curve that coincides with the vertical average life line on the mortality charts. The rectangle formed by such a curve has been referred to hereinbefore. Types II to VII diverge more and more from the straight line and continue on to longer lives. Type VII maintains the high-

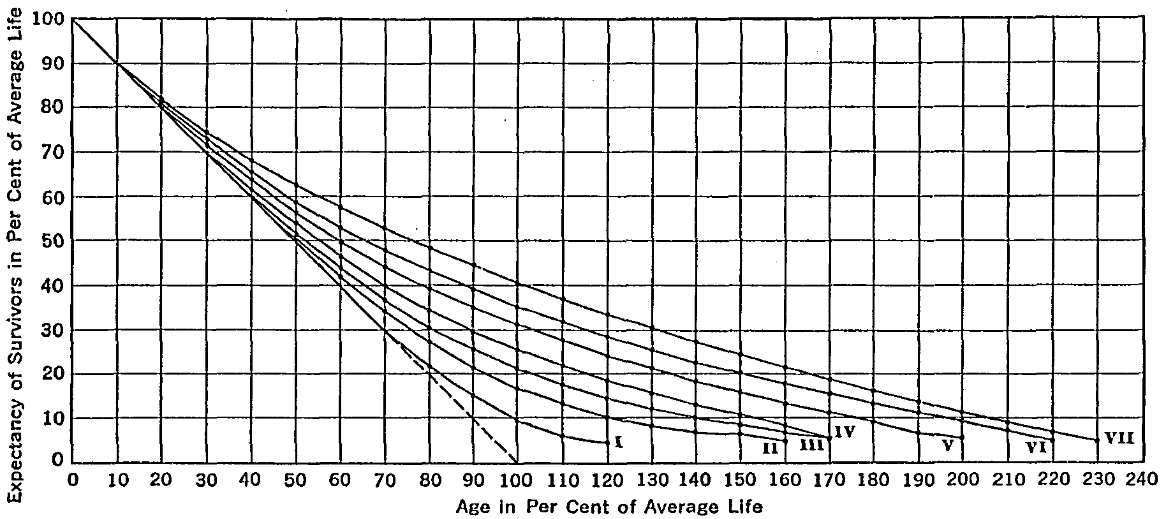


Figure 72. Life Expectancies of Survivors for Seven Graduated Type Mortality Curves

est expectancy at all ages and continues on to 230% of average life.

Types of Probable Life Curves.—The seven type probable life curves are obtained directly from the seven type expectancy curves by adding ages and corresponding expectancies in accordance with the formula:

Probable Life at Any Age = Age + Expectancy at That Age

The data for the seven type probable life curves are given in Table 46 and the corresponding curves in Figure 73. On this figure are plotted also the probable life curves obtained directly

Table 46. Probable Lives in Per Cent of Average Life for Seven Graduated Type Mortality Curves

Age in Per Cent of Average Life	Types						
	I	II	III	IV	V	VI	VII
0	99.9	101.5	100.1	99.9	99.8	100.0	99.9
10	99.9	101.5	100.1	99.9	100.0	100.3	100.4
20	99.9	101.5	100.1	100.1	100.7	101.2	102.1
30	99.9	101.5	100.3	100.8	101.9	102.9	104.8
40	99.9	101.5	100.7	102.0	103.9	105.4	108.5
50	100.0	101.7	101.7	103.9	106.7	108.8	112.7
60	100.1	102.4	103.6	106.7	110.3	112.9	117.6
70	100.6	104.0	106.4	110.3	114.5	117.7	122.9
80	101.8	106.9	110.3	114.8	119.4	123.1	128.5
90	104.4	111.1	115.2	120.0	124.8	128.9	134.5
100	108.9	116.7	121.0	125.9	130.7	135.1	140.7
110	115.3	123.3	127.4	132.2	137.1	141.7	147.1
120	125.0	130.6	134.5	139.0	143.8	148.5	153.7
130		138.7	142.0	146.1	150.7	155.5	160.5
140		147.6	150.0	153.5	157.9	162.8	167.4
150		157.0	158.2	160.9	165.4	170.2	174.4
160		165.0	166.6	168.1	173.0	177.8	181.5
170			175.0	175.0	180.9	185.5	188.7
180					188.8	193.4	196.0
190					196.5	201.4	203.3
200					205.0	209.4	210.8
210						217.5	218.2
220						225.0	225.9
230							235.0

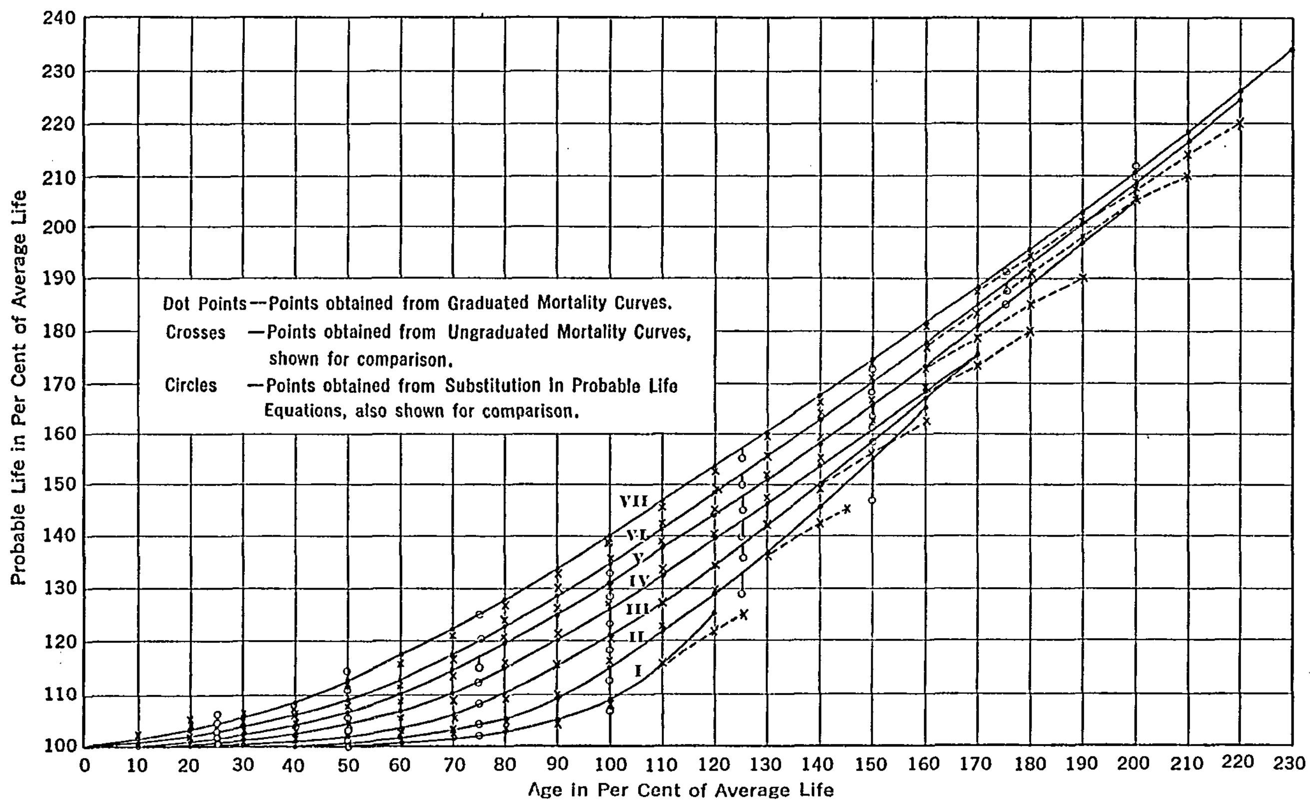


Figure 73. Probable Life Curves of Survivors for Seven Graduated Type Mortality Curves

from the ungraduated type mortality curves. That no wide variations exist between these probable life values and those calculated from the graduated type mortality curves resulting from the frequency equations is evident and shows that no serious discrepancies have been introduced. The closeness of fit here shown is further proof that the Pearsonian frequency equations represent the seven type distribution curves with satisfactory accuracy. These probable life curves, unlike the expectancy curves, are increasing functions with age. This, of course, is due to the addition of a straight-line function with age. Since the expectancy curves have a tendency to drop during the early period by an amount almost equal to age, the addition of an amount equal to age, therefore, compensates and maintains the early portion of the probable life curve. The probable life curve is thus identical to the expectancy curve except that it has been rotated 45° in the counter-clockwise direction.

Equations of Type Curves.—From an inspection of the expectancy and probable life curves it is apparent that the probable life curves appear to represent natural law curves more nearly than the expectancy curves. The class of curves which the probable life curves appear to represent are parabolic, given by the expression $y = ax^b$. Such curves pass through the origin, and as one of the variables increases the other increases also. By placing the origin at 0, 100, and adding 100 to all values of y, this expression for the parabola can be adapted to represent the probable life curves, thus:

$$\text{Probable Life} = 100 + ax^b$$

where a is a coefficient, b an exponent, and the x the age in per cent of average life.

To test the assumption that the probable life curves belong to the parabolic class, it is only necessary to plot the coordinates on log-log paper and note whether the points lie along a straight

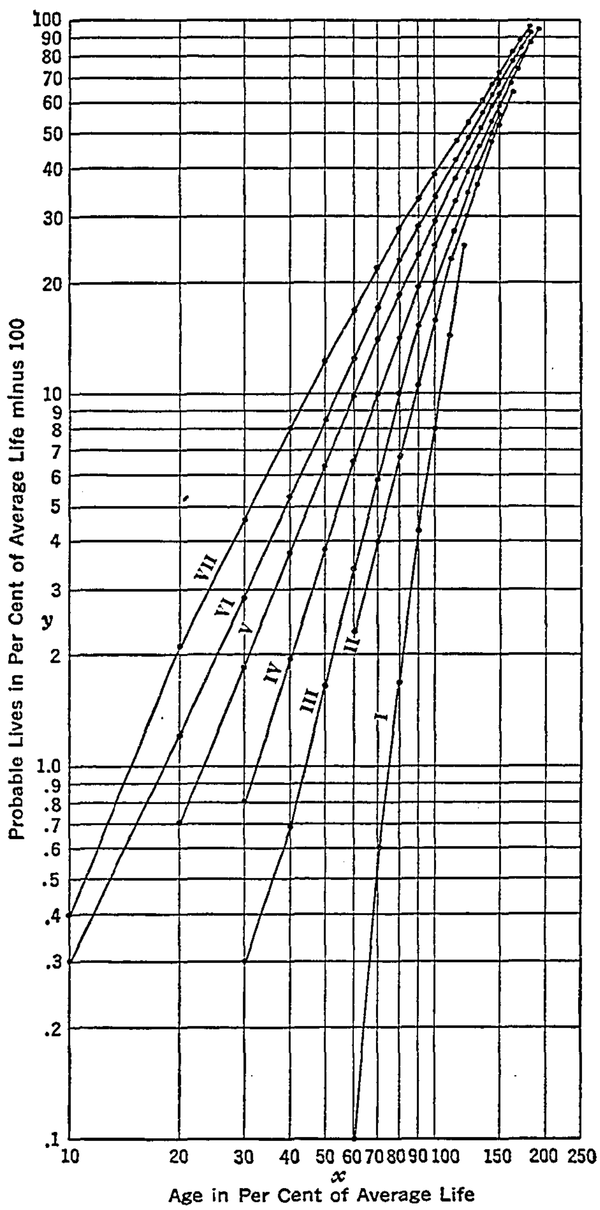


Figure 74. Type Probable Life Curves Plotted on Log-Log Paper

line. By taking the logarithms of each side of the equation $y = ax^b$ there results:

$$\log y = \log a + b \log x$$

which is the equation of a straight line with "log a" as intercept, and b as slope. Therefore, by plotting the logarithms of the numbers on standard coordinate paper, or by plotting the numbers themselves on log-log paper, the accuracy of the assumption that the probable life curves are simple parabolas can be determined. Figure 74 shows the seven probable life curves plotted on log-log paper. The plotted values of y have, however, all been reduced by 100, which quantity is added again in the final expression as already noted. These data plot into fairly straight lines, and the expressions given below have been derived from these lines.

Derivation of Expressions.—The straight line which best fitted the main part of each curve was selected and then two sets of values were read for each line. For curve I, for example, the values to represent the straight line were:

x	y	$\log x$	$\log y$
120	25	2.07918	1.39794
100	8	2.0000	0.90309

Since the expression $y = ax^b$ can be written

$$\log y = \log a + b \log x$$

the following substitutions can be made:

$$\begin{aligned} 1.39794 &= \log a + b \times 2.07918 \\ 0.90309 &= \log a + b \times 2.0000 \\ \hline 0.49485 &= \phantom{\log a + {}} b \times 0.07918 \end{aligned}$$

and by subtracting the second from the first the value for b can be obtained thus:

$$b = \frac{0.49485}{0.07918} = 6.2497$$

By substituting for b in the second expression, thus:

$$0.90309 = \log a + 6.2497 \times 2.0000$$
$$\log a = 0.90309 - 6.2497 \times 2.0000$$
$$= -11.59631$$
$$= -12 + 0.40369$$

and $$a = 0.000000000002533$$

Therefore the expression for probable life becomes:

$$\text{Probable Life} = 100 + 0.000000000002533 \times x^{6.2497}$$

And since Probable Life = Age + Expectancy

$$\text{Expectancy} = \text{Probable Life} - \text{Age}$$

or $$\text{Expectancy} = 100 - x + 0.000000000002533 \times x^{6.2497}$$

In like manner the constants for the expressions of each of the other type curves were obtained. The constants for the seven types are tabulated in Table 47 and plotted in Figure 75.

The plotted curves in Figure 75 show that the changes from one type to the other are quite gradual. It will also be noted that as the value of a goes up b comes down, and vice versa.

Significance of Parabolic Equations.—The manner in which the equations fit the data from which they were derived is also shown in Figure 73. The points shown as dots are the values of probable life obtained from the graduated mortality data, and the points shown as circles are values obtained by substitution in the parabolic equations. The check is good enough to permit the statement that the probable life curves represent natural laws expressible by means of a parabola added to a constant.

This finding is of considerable note as it expresses a definite relation between probable life and age. The interesting thing brought out is the fact that the older a unit becomes the more age it can be expected to acquire. This, of course, is to be expected, for every hazard that a unit succeeds in surviving in its early life improves its chances for continuing in service

In early life the increase is small and slow, but in later life the increase is very rapid and becomes large in amount.

Relation of Types to Kinds of Property.—Reference has already been made in Chapters 3 and 4 to the classification of

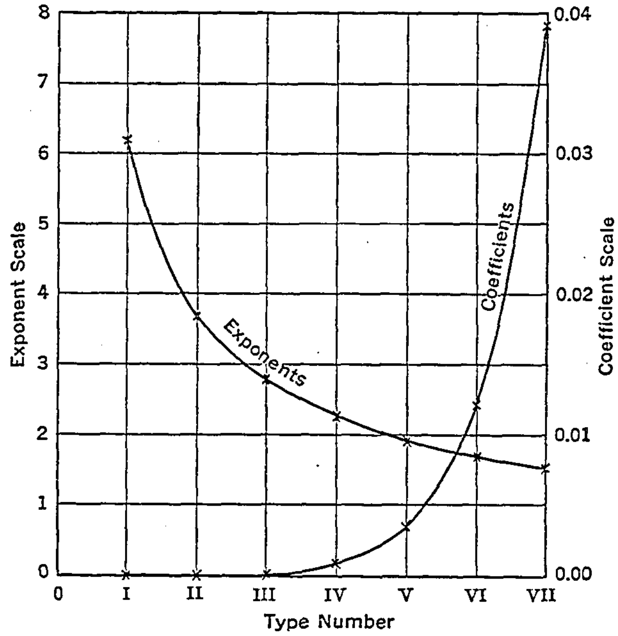


Figure 75. Variation in Constants in Probable Life Equations for Seven Type Curves

Table 47. Constants for Parabolic Equations for Seven Type Probable Life Curves

Types	Coefficient (a)	Exponent (b)
I...........	0.000000000002533	6.2497
II...........	0.0000005233	3.6971
III...........	0.00004767	2.7964
IV...........	0.0005422	2.3212
V...........	0.003332	1.965
VI...........	0.01228	1.7202
VII...........	0.03953	1.500

the 52 property groups into the seven type classes. It is of interest to note here briefly the significance of a group being in one of the type classes as concerns expectancy and probable life. Property classified into Types I, II, and III such as ties and railroad equipment, maintains a low expectancy throughout life. This means that more of the units of the group live out the average life, and few units live longer than average. Property such as waterwork equipment, telephone, telegraph, and electric equipment which falls largely into the upper classes, Types IV to VII, however, maintains high expectancies reaching in extreme cases probable lives equal to 230% of average life.

Expectancy at Average Life.—It was pointed out in an earlier chapter that almost 50% of the units of a group of property units will be in service as the average life age is reached. Naturally, there will also remain at the average life age a life expectancy for the units remaining in service. This expectancy, however, varies for different groups of property, ranging all the way from less than 10% to more than 50% of average life, depending upon the shape of the mortality curve. For the seven type curves the expectancies at average life are as shown in Table 48.

The percentages for the type curves indicate the range as well as the values for various shape curves. These expectancies plot into the graph shown in Figure 76. The change from one type to the other is gradual showing that the types were well selected. It is also of interest to note that the extension of the curve would intersect the origin. The type represented by the origin, since it represents 0 expectancy, undoubtedly is the case of a vertical mortality curve, coincident with the average life line. Such a mortality curve would have no area beyond the average life line, and hence 0 expectancy. Its expectancy curve would be a straight line from 100% at age 0 to 0 expectancy at 100% average life.

The value of the expectancy at average life is determined largely by the magnitude of the area *A* in Figure 77. If this area is small as in (a), the expectancy is small; and if the area is large as in (b), the expectancy will be large. It is apparent

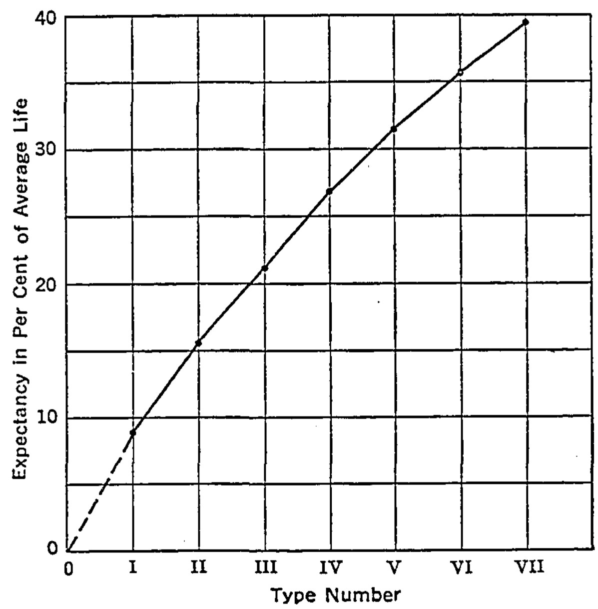


Figure 76. Variation of Life Expectancy at Average Life for Seven Type Mortality Curves

Table 48. Life Expectancies at Average Life for Seven Type Mortality Groups

Type Number	Life Expectancy at Average Life in Per Cent of Average Life
I	8.65
II	15.36
III	20.96
IV	26.63
V	31.25
VI	35.66
VII	39.76

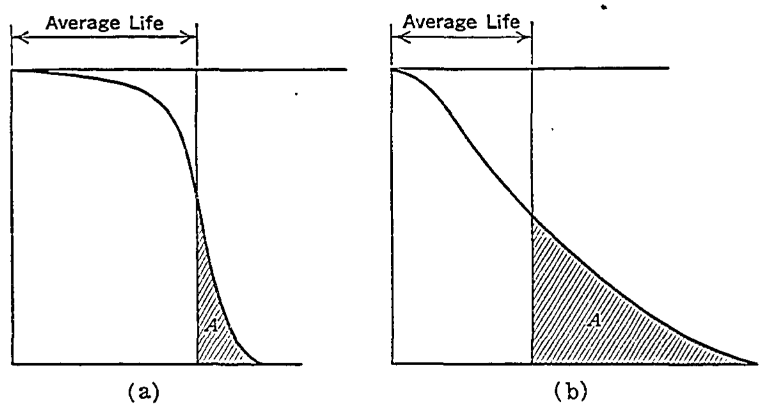


Figure 77. Effect of Slope of Mortality Curve upon Magnitude of Area Beyond Average Life Line

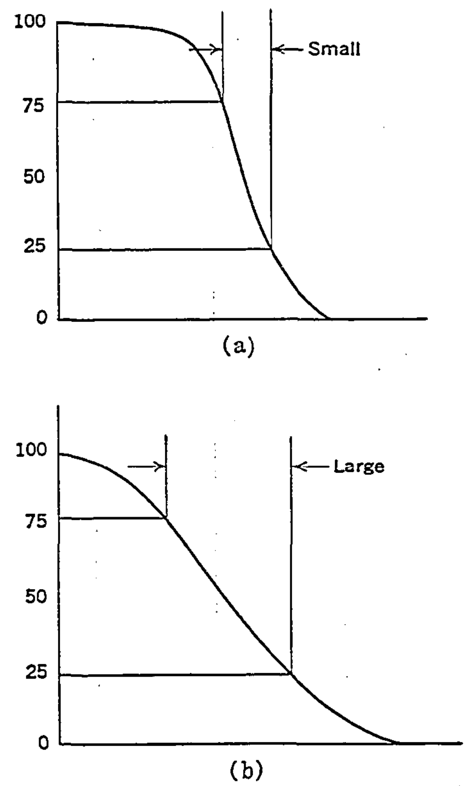


Figure 78. Method of Expressing Slope of Mortality Curve by Time Interval Between 3/4 and 1/4 Quartiles

Table 49. Values of Expectancy at Average Life, and Time Interval Between ¾ and ¼ Quartiles for 52 Property Groups

Mortality Chart Number	Per Cent of Average Life at		Per Cent of Average Life Between ¾ and ¼ Quartiles	Expectancy at Average Life in Per Cent of Average Life
	¾ Quartile	¼ Quartile		
1	55	140	85	47
2	60	130	70	50
3	65	125	60	40
4	80	117	37	26
5	77	124	47	24
6	60	130	70	43
7	95	116	21	14
8	60	136	76	43
9	50	145	95	64
10	60	140	80	52
11	70	126	56	35
12	62	136	74	37
13	62	138	76	40
14	72	125	53	32
15	70	118	48	37
16	75	126	51	30
17	65	134	69	36
18	58	140	82	53
19	48	150	102	73
20	68	120	52	46
21	65	128	63	36
22	55	140	85	48
23	72	122	50	35
24	74	125	51	27
25	65	127	62	42
26	72	125	53	36
27	75	125	50	32
28	65	134	69	36
29	87	114	27	14
30	78	125	47	25
31	70	124	54	36
32	65	135	70	44
33	76	119	43	26
34	84	116	32	21
35	86	115	29	17
36	88	115	27	17
37	85	116	31	17
38	84	116	32	17
39	85	108	23	12
40	79	118	39	22
41	92	109	17	7
42	94	108	14	7
43	86	112	26	14
44	89	113	24	12
45	84	114	30	15
46	86	114	28	15
47	86	111	25	14
48	85	110	25	28
49	87	114	27	13
50	81	113	32	30
51	80	115	35	20
52	87	113	26	12

that a steep mortality curve as in (a) will have a small A area, and that a curve with a gradual slope as in (b) will have a large area. If, therefore, a function can be found to represent this slope it is likely that a relation between slope and expectancy at average life can be obtained.

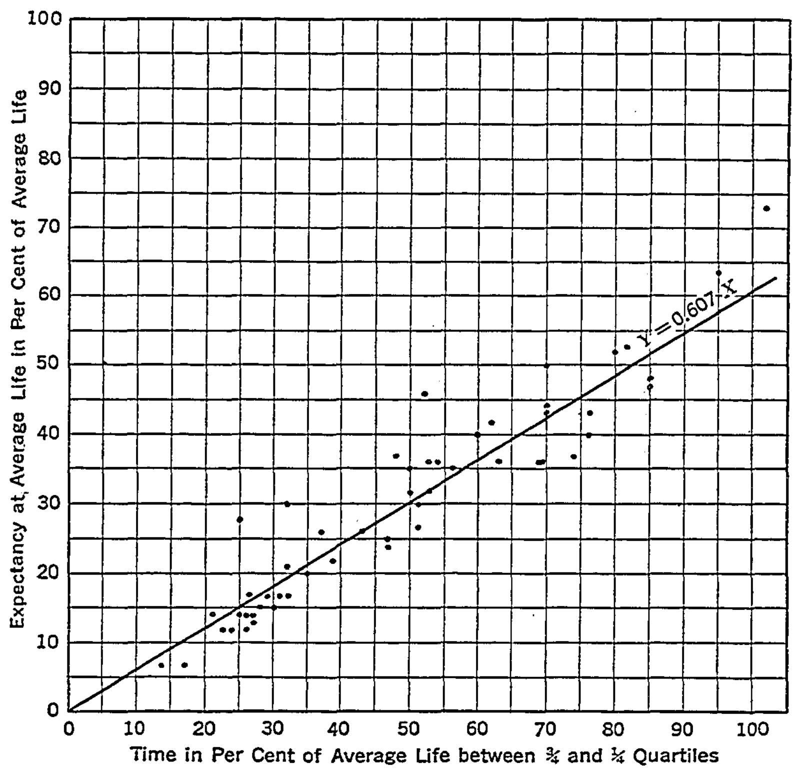


Figure 79. Relation Between Expectancy at Average Life and Time Interval Between 3/4 and 1/4 Quartiles

The slope of a mortality curve can be fairly well indicated or expressed by the time interval between the ¾ and ¼ quartiles. The quartiles are the 75% and 25% points on the mortality curve. If the time interval between these quartiles is small as in Figure 78 (a), the curve is steep; and if it is large as in Figure 78 (b), the curve is not steep.

To test the generalization proposed, Table 49 was prepared. This table gives the time interval between the ¾ and ¼ quartiles, and the expectancy at average life, for each of the 52 groups of property. To show the relationship suggested, the time interval between quartiles was plotted against life expectancy in Figure 79. The plotted points, although slightly

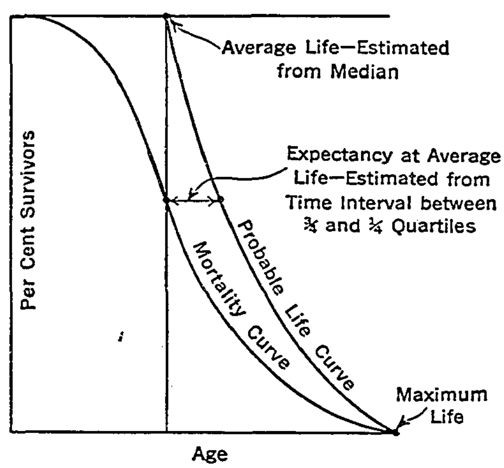


Figure 80. Method of Sketching in Probable Life Curve from Estimates of Average Life, Expectancy at Average Life, and Known Value of Maximum Life

spread, indicate a very pronounced relation. This relation can be expressed by the equation,

$$Y = 0.607\ X,$$

where Y is the expectancy at average life, and X is the time interval in per cent of average life between the ¾ and ¼ quartiles. The line goes through the origin showing that when the mortality curve is vertical there can be no expectancy at average life. For large intervals of time between the quartiles the expectancy runs as high as 50% and in extreme cases over 70%.

The above relation thus provides a means of estimating the expectancy at average life for any given mortality curve. A method of estimating the average life by use of the median has already been presented. The maximum life for any given mortality curve is known. These three points are on the probable life curve as drawn on the mortality charts. (See Figure 80.) With three points known, the probable life curve can be sketched in with a fair degree of accuracy.

CHAPTER 8

ANNUAL RENEWALS

Replacement of Original Units.—This subject has already been extensively treated in Chapter 4. Suffice it only to reiterate briefly that when a group of units is placed in service simultaneously, a few go out of service almost immediately, some remain in service a few years longer, others stay in service up to and around the expected average life, while approximately half of the original units give service beyond the expected life of the average unit; in fact, some may stay in service beyond 200% of average life. When the number that go out of service each year are plotted against the age reached at retirement, the well-known distribution or frequency curve results.

Discussion in Chapter 4 included the derivation of a Pearsonian frequency equation for each of the seven type frequency curves resulting from the seven type mortality curves developed in Chapter 3. From these empirical equations a new set of frequency curves was obtained. Further reference will be made to these curves in the paragraphs that follow.

Renewals of Replacements.—In a going concern the original units removed or retired each year must be replaced at once by new units. The new units in turn will likewise have to be replaced by others from year to year in the same proportion as the original units of the same length of service. Hence the total renewals for any one year will be in excess of the renewals of the original units, as they will include renewals of replacements as well. In fact, after the first life cycle the annual renewals will be made up entirely of renewals of replacements as all the original units will have been retired.

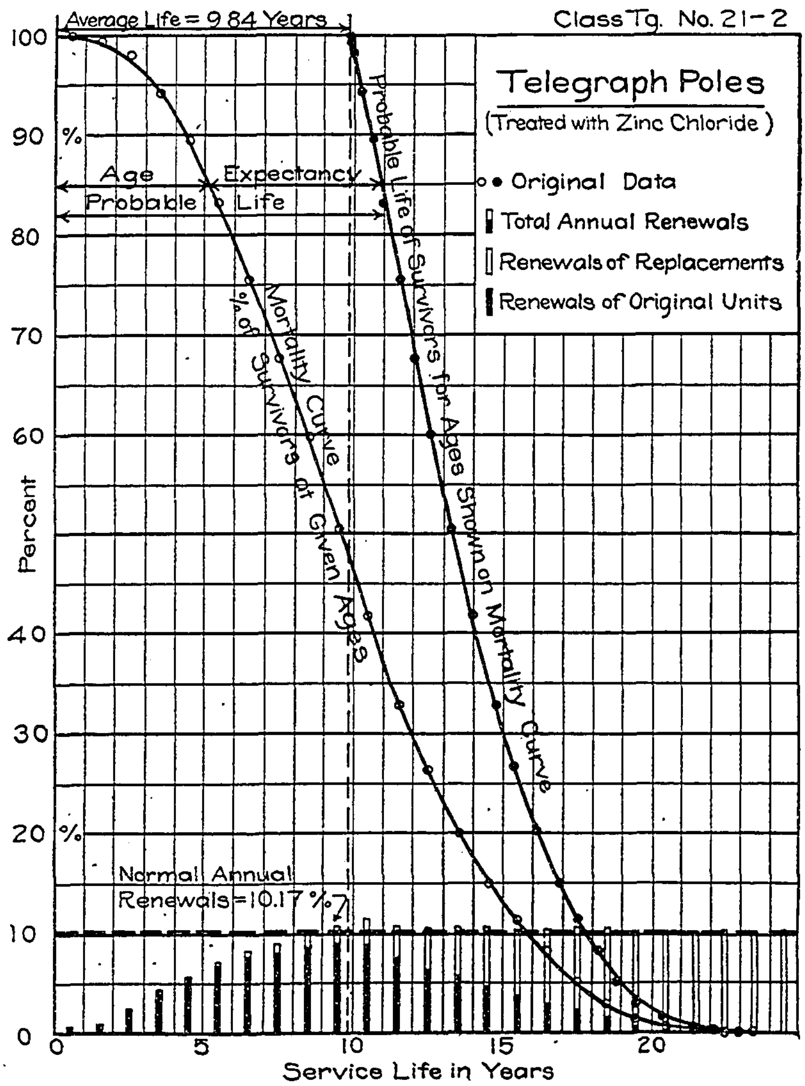


Figure 81. Mortality Chart of Telegraph Poles Showing Method of Indicating Renewals of Original Units, Renewals of Replacements, and Total Annual Renewals

The renewals of replacements for the 52 property groups are shown on the mortality charts by extensions to the vertical bars for replacements of original units (Figure 81). The solid portion of the bars represents replacement of original units and the hollow portion the renewals of replacements. The total height of bar, therefore, represents the total annual renewals.

Total Annual Renewals.—From the above it follows that the total annual renewals are given by the expression:

Total Annual Renewals = Replacements of Original Units + Renewals of Replacements

The replacements of the original units are fixed by the distribution curve for the particular property group or may be determined by the use of the frequency formula for the given property. To compute the renewals of replacements, however, is a more tedious process and a number of assumptions must be made. These are:

1. That the replacements of the original units are made at the middle of the age interval.
2. That the replacements begin to go out of service immediately after being placed in service.
3. That the renewals of replacements are governed by the same frequency distribution as that governing the original units.
4. That the annual renewal rates or "annual renewal multipliers" are based on the sum of one-half of the renewal rates of adjacent intervals. This assumption is the basis of the computation of the renewal multipliers discussed below.

A typical illustration of the calculations based on these assumptions is given in the next paragraph.

Method of Computing Annual Renewals.—The method used in calculating the total annual renewals for each of the

52 property groups will be illustrated by referring to Table 50. The renewal calculations shown in this table are those for Type VII of the type curves developed in Chapters 3 and 4. Across the top of the table appear the intervals for one life cycle. On the extreme left is a vertical time scale beginning at the time the property group is placed in service. The third horizontal column of figures gives the per cent replacements of the original units thus: 0.46, 1.94, 3.38, 4.64, 5.68, 6.50, 7.07, 7.43, 7.56, 7.51, etc., to end of property life. The total of these per cents should, of course, equal 100% and is so indicated on the extreme right end of line. The second horizontal column gives the so-called "annual renewal multipliers," thus 0.23, 1.20, 2.66, 4.01, 5.16, 6.09, 6.785, 7.25, 7.495, 7.535, etc. These also total 100% or unity. The values of these multipliers are obtained by averaging the preceding and succeeding replacement percentages of original units, thus:

$$\frac{0 + 0.46}{2} = 0.23; \frac{0.46 + 1.94}{2} = 1.20; \frac{1.94 + 3.38}{2} = 2.66;$$
$$\frac{3.38 + 4.64}{2} = 4.01; \text{etc.}$$

As stated above, this procedure is based on the assumption that the replacements of original units are made during the regular age interval. The replacements for each year can therefore be assumed to be concentrated at the middle of the interval. The age interval for the renewals of replacements would thus be shifted ½ interval back in time. Since only ½ of an interval remains in the interval (0–10) after the renewals of original units are made, only ½ of the (0–10) renewal rate is used; hence, $\frac{0.46}{2} = 0.23$. In the interval (10–20) the remaining ½ rate for the first interval and half of the rate for the second interval are effective; hence $\frac{0.46 + 1.94}{2} = 1.20$, is the renewal rate for the replacements during the period (10–20). In like manner the renewals of

Table 50. Total Annual Renewal Calculations for Graduated Type Frequency and Mortality Group VII

(Calculations are shown for three maximum life cycles.)

Age in % of Avge. Life	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150	150–160	160–170	170–180	180–190	190–200	200–210	210–220	220–230	230–240	240–250	
Annual Renewal Multipliers	0.230	1.200	2.660	4.010	5.160	6.090	6.785	7.250	7.495	7.535	7.400	7.100	6.655	6.100	5.455	4.745	4.005	3.260	2.535	1.855	1.250	0.745	0.360	0.110	0.010	100.00
Renewals of Original Units %	0.460	1.940	3.380	4.640	5.680	6.500	7.070	7.430	7.560	7.510	7.290	6.910	6.40	5.800	5.110	4.380	3.630	2.890	2.180	1.530	0.970	0.520	0.200	0.020	0.000	100.00
10	0.001	0.006	0.012	0.018	0.024	0.028	0.031	0.033	0.035	0.035	0.034	0.033	0.031	0.028	0.025	0.022	0.018	0.015	0.012	0.009	0.006	0.003	0.002	0.000	0.000	0.461
20	0.000	0.005	0.023	0.052	0.078	0.101	0.119	0.132	0.141	0.146	0.147	0.144	0.139	0.130	0.119	0.106	0.093	0.078	0.064	0.049	0.036	0.024	0.015	0.007	0.002	1.951
30	0.004	0.000	0.007	0.041	0.091	0.137	0.177	0.208	0.232	0.248	0.257	0.258	0.253	0.243	0.228	0.209	0.187	0.162	0.137	0.112	0.087	0.063	0.043	0.026	0.012	3.422
40	0.017	0.005	0.000	0.011	0.057	0.127	0.191	0.246	0.290	0.323	0.315	0.357	0.359	0.352	0.338	0.317	0.290	0.260	0.226	0.191	0.155	0.121	0.088	0.060	0.036	4.762
50	0.014	0.021	0.007	0.001	0.014	0.071	0.158	0.238	0.307	0.362	0.403	0.431	0.445	0.448	0.440	0.422	0.396	0.363	0.324	0.282	0.238	0.194	0.151	0.110	0.074	5.944
60	0.087	0.052	0.025	0.008	0.001	0.016	0.084	0.186	0.280	0.360	0.425	0.473	0.506	0.523	0.526	0.517	0.496	0.464	0.426	0.381	0.331	0.279	0.228	0.177	0.129	6.980
70	0.146	0.098	0.058	0.028	0.009	0.001	0.018	0.094	0.209	0.315	0.405	0.478	0.533	0.569	0.588	0.591	0.581	0.557	0.522	0.479	0.428	0.372	0.314	0.258	0.199	7.848
80	0.218	0.159	0.107	0.064	0.031	0.009	0.001	0.020	0.103	0.228	0.344	0.443	0.523	0.583	0.623	0.644	0.647	0.635	0.610	0.571	0.524	0.468	0.408	0.314	0.280	8.587
90	0.299	0.233	0.170	0.115	0.068	0.033	0.010	0.001	0.021	0.110	0.244	0.368	0.474	0.559	0.623	0.665	0.688	0.691	0.679	0.652	0.611	0.560	0.501	0.435	0.368	9.178
100	0.387	0.315	0.245	0.179	0.121	0.072	0.035	0.011	0.001	0.022	0.116	0.257	0.387	0.498	0.588	0.655	0.700	0.724	0.728	0.715	0.686	0.643	0.599	0.527	0.458	9.659
110	0.476	0.402	0.327	0.254	0.186	0.126	0.075	0.036	0.011	0.001	0.023	0.120	0.267	0.402	0.518	0.611	0.681	0.727	0.752	0.756	0.743	0.712	0.668	0.612	0.547	10.033
120	0.562	0.488	0.412	0.336	0.261	0.191	0.129	0.077	0.037	0.011	0.001	0.024	0.124	0.274	0.413	0.531	0.627	0.699	0.746	0.772	0.775	0.762	0.731	0.685	0.628	10.296
130	0.638	0.571	0.497	0.419	0.341	0.265	0.191	0.131	0.078	0.038	0.012	0.001	0.024	0.125	0.278	0.420	0.540	0.637	0.710	0.759	0.784	0.789	0.774	0.743	0.697	10.465
140	0.703	0.644	0.576	0.501	0.423	0.344	0.268	0.198	0.132	0.079	0.038	0.012	0.001	0.024	0.127	0.281	0.423	0.545	0.643	0.716	0.765	0.791	0.795	0.781	0.750	10.558
150	0.750	0.703	0.645	0.576	0.502	0.423	0.345	0.268	0.196	0.132	0.079	0.038	0.012	0.001	0.024	0.127	0.281	0.424	0.545	0.644	0.717	0.766	0.792	0.796	0.782	10.568
160	0.779	0.747	0.700	0.642	0.574	0.499	0.421	0.343	0.267	0.195	0.132	0.078	0.038	0.012	0.001	0.024	0.126	0.280	0.422	0.543	0.641	0.714	0.763	0.789	0.793	10.522
170	0.786	0.772	0.740	0.694	0.636	0.569	0.495	0.418	0.340	0.264	0.193	0.130	0.078	0.038	0.011	0.001	0.024	0.125	0.277	0.418	0.538	0.635	0.708	0.756	0.782	10.428
180	0.772	0.776	0.762	0.731	0.685	0.628	0.562	0.489	0.413	0.336	0.261	0.191	0.129	0.077	0.037	0.011	0.001	0.024	0.124	0.274	0.413	0.531	0.627	0.699	0.747	10.300
190	0.736	0.761	0.765	0.751	0.721	0.675	0.619	0.554	0.481	0.406	0.331	0.257	0.188	0.127	0.076	0.037	0.011	0.001	0.023	0.122	0.270	0.407	0.524	0.618	0.689	10.150
200	0.678	0.725	0.749	0.753	0.740	0.710	0.665	0.610	0.545	0.474	0.400	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	9.998
210	0.601	0.669	0.715	0.739	0.743	0.730	0.700	0.656	0.602	0.538	0.468	0.305	0.321	0.250	0.183	0.123	0.073	0.036	0.011	0.001	0.023	0.118	0.262	0.395	0.509	9.861
220	0.504	0.595	0.662	0.708	0.732	0.735	0.722	0.693	0.650	0.595	0.532	0.463	0.391	0.318	0.247	0.181	0.122	0.073	0.035	0.011	0.001	0.022	0.117	0.260	0.391	9.760
230	0.390	0.502	0.592	0.660	0.705	0.729	0.733	0.720	0.690	0.647	0.593	0.530	0.461	0.389	0.317	0.246	0.180	0.122	0.072	0.035	0.011	0.001	0.022	0.117	0.259	9.723
240	0.259	0.391	0.503	0.594	0.662	0.707	0.731	0.735	0.721	0.692	0.649	0.595	0.532	0.463	0.390	0.318	0.247	0.181	0.122	0.073	0.035	0.011	0.001	0.022	0.117	9.751
250	0.119	0.263	0.396	0.510	0.602	0.670	0.716	0.740	0.744	0.731	0.702	0.658	0.603	0.539	0.469	0.390	0.322	0.250	0.183	0.123	0.074	0.036	0.011	0.001	0.023	9.881
260	0.023	0.120	0.265	0.400	0.515	0.608	0.677	0.723	0.748	0.752	0.738	0.708	0.664	0.609	0.544	0.474	0.400	0.325	0.253	0.185	0.125	0.074	0.036	0.011	0.001	9.978
270	0.001	0.023	0.120	0.267	0.402	0.518	0.611	0.681	0.728	0.752	0.756	0.743	0.713	0.668	0.612	0.548	0.476	0.402	0.327	0.254	0.186	0.125	0.075	0.036	0.011	10.035
280	0.011	0.001	0.023	0.121	0.268	0.403	0.519	0.613	0.683	0.729	0.754	0.758	0.745	0.714	0.669	0.614	0.549	0.477	0.403	0.328	0.255	0.187	0.126	0.075	0.036	10.061
290	0.036	0.011	0.001	0.023	0.121	0.268	0.404	0.520	0.614	0.684	0.730	0.755	0.759	0.745	0.715	0.670	0.615	0.550	0.478	0.404	0.328	0.255	0.187	0.126	0.075	10.074
300	0.075	0.036	0.011	0.001	0.023	0.121	0.268	0.404	0.520	0.613	0.683	0.730	0.755	0.759	0.745	0.715	0.670	0.615	0.550	0.478	0.404	0.328	0.255	0.187	0.126	10.072
310	0.126	0.075	0.036	0.011	0.001	0.023	0.121	0.268	0.403	0.519	0.612	0.682	0.729	0.754	0.758	0.744	0.714	0.669	0.613	0.549	0.477	0.403	0.328	0.255	0.187	10.057
320	0.186	0.126	0.075	0.036	0.011	0.001	0.023	0.121	0.267	0.403	0.518	0.612	0.681	0.728	0.753	0.757	0.743	0.713	0.668	0.613	0.548	0.477	0.402	0.327	0.255	10.044
330	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.267	0.402	0.518	0.611	0.681	0.727	0.752	0.756	0.742	0.712	0.668	0.612	0.547	0.476	0.402	0.327	10.031
340	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.402	0.517	0.610	0.679	0.726	0.751	0.755	0.741	0.711	0.666	0.611	0.546	0.475	0.401	10.014
350	0.400	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.545	0.474	10.001
360	0.474	0.400	0.326	0.253	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.724	0.749	0.753	0.739	0.709	0.665	0.610	0.545	9.993
370	0.545	0.474	0.400	0.326	0.253	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.608	0.678	0.724	0.749	0.753	0.739	0.709	0.665	0.609	9.991
Renewals	0.461	1.951	3.422	4.762	5.944	6.980	7.848	8.587	9.178	9.650	10.033	10.296	10.465	10.558	10.568	10.522	10.428	10.300	10.150	9.998	9.861	9.760	9.723	9.751	9.881	
Renewals	9.978	10.035	10.061	10.074	10.072	10.057	10.044	10.031	10.014	10.001	9.993	9.991														

380	0.609	0.515	0.474	0.400	0.326	0.253	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.724	0.749	0.753	0.739	0.709	0.665	9.992
390	0.665	0.610	0.545	0.474	0.400	0.326	0.253	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.725	0.749	0.753	0.739	0.709	9.994
400	0.709	0.665	0.610	0.545	0.474	0.400	0.326	0.253	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.724	0.749	0.753	0.739	9.993
410	0.740	0.710	0.666	0.610	0.515	0.474	0.400	0.326	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.725	0.750	0.754	10.001
420	0.751	0.740	0.710	0.666	0.610	0.546	0.474	0.400	0.326	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	10.003
430	0.750	0.751	0.741	0.711	0.666	0.610	0.516	0.475	0.401	0.326	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.610	0.679	0.726	10.009
440	0.726	0.750	0.751	0.741	0.711	0.666	0.610	0.546	0.475	0.401	0.326	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.610	0.679	10.009
450	0.679	0.726	0.750	0.754	0.741	0.711	0.666	0.611	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.402	0.517	0.610	10.012
460	0.610	0.679	0.726	0.750	0.754	0.741	0.711	0.666	0.611	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.517	10.011
470	0.516	0.610	0.679	0.725	0.750	0.754	0.741	0.711	0.666	0.610	0.516	0.475	0.401	0.326	0.254	·0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	10.008
480	0.401	0.516	0.610	0.679	0.726	0.750	0.751	0.741	0.711	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	10.009
490	0.266	0.401	0.516	0.609	0.679	0.726	0.750	0.754	0.741	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	10.007
500	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.751	0.741	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	10.006
510	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.545	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	10.002
520	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.546	0.474	0.401	0.326	0.251	0.186	0.125	0.075	0.036	0.011	10.004
530	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.251	0.186	0.125	0.075	0.036	10.005
540	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.516	0.474	0.401	0.326	0.254	0.185	0.125	0.075	10.003
550	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.546	0.474	0.401	0.326	0.254	0.185	0.125	10.003
560	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.185	10.004
570	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.678	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.546	0.474	0.401	0.326	0.254	10.002
580	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	10.005
590	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0 120	0.266	0.401	0.516	0.609	0.679	0.726	0.750	0.754	0.741	0.710	0.666	0.610	0.546	0.475	0.401	10.007
600	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.741	0.710	0.666	0.610	0.546	0.475	10.006
610	0.475	0.401	0.320	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	10.005
620	0.546	0.475	0.401	0.326	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	10.005
630	0.610	0.546	0.475	0.401	0.326	0.251	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	10.004
640	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	10.005
650	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	10.004
660	0.741	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.510	0.609	0.679	0.725	0.750	0.754	10.006
670	0.751	0.740	0.710	0.666	0.610	0.516	0.475	0.401	0.326	0.251	0.185	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	10.004
680	0.750	0.751	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	0.725	10.005
690	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.251	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	0.679	10.005
700	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	0.609	10.005
710	0.609	0.679	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	0.516	10.005
720	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	0.401	10.004
730	0.401	0.516	0.609	0.679	0.725	0.750	0.751	0.740	0.710	0.666	0.610	0.516	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	0.266	10.004
740	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	0.120	10.004
750	0.120	0.266	0.401	0.516	0.609	0.679	0.725	0.750	0.754	0.740	0.710	0.666	0.610	0.546	0.475	0.401	0.326	0.254	0.186	0.125	0.075	0.036	0.011	0.001	0.023	10.004
760	0.023	0.120	0.266	0.401	0.516																					10.005
770		0.023	0.120	0.266	0.401																					10.005
780			0.023	0.120	0.266																					10.005
790				0.023	0.120																					10.005
800					0.023																					10.005
Renewals													9.992	9.994	9.993	10.001	10.003	10.009	10.009	10.012	10.011	10.008	10.009	10.007	10.006	
Renewals	10.002	10.004	10.005	10.003	10.003	10.004	10.002	10.005	10.007	10.006	10.005	10.005	10.004	10.005	10.004	10.006	10.004	10.005	10.005	10.005	10.005	10.004	10.004	10.004	10.004	
Renewals	10.005	10.005	10.005	10.005	10.005																					

replacements in each succeeding period are governed by the sum of ½ of the renewal rates for each of the intervals included.

Each renewal is computed by multiplying the per cent of total units replaced in any interval by each of the renewal multipliers. For example, the replacements of original units during the first interval (0–10) is 0.46%. As soon as these replacements are made the new units in turn will begin to go out of service; in fact, in the last half of the (0–10) interval, the 0.46 units will undergo renewals given by $0.46 \times 0.0023053 = 0.001\%$ units, so that the total units which will have to be distributed over the life cycle is 0.461 instead of 0.46. This will be further explained below.

Since the renewals are assumed to be governed by the same retirement rates as the original units (except for the change in interval already noted) the renewals of replacements for the interval (10–20) will be $0.461 \times 0.012 = 0.006$; for the interval (20–30) will be $0.461 \times 0.0266 = 0.012$; for the interval (30–40) will be $0.461 \times 0.0401 = 0.018$, etc. When the multiplications with each of the multipliers have been carried out the life cycle has been run and all of the 0.461% units will have been distributed over the life cycle in accordance with the distribution of the original units. The sum of all the renewals appearing in this horizontal line totals 0.461, the original replacement plus the additional 0.001 which is due to renewals of renewals in the year of placement and will be further explained in the following paragraphs.

The total units replaced in any one age interval may thus be made up of replacement of original units, renewals of replacements, and as will be further shown below, of renewals of renewals in the year of placement. The total annual renewal units during any interval in the first life cycle comprise all the renewals in a vertical column from the top of table down to the broken line. In the succeeding life cycles the total annual renewals comprise all the renewals in a vertical column between two adjacent broken lines. As an example, note the renewals made in the age interval (20–30). There will be found the replacements of the original units of 3.38%, the renewals of the first year replacements of 0.012%, the renewals of the second year replacements of 0.023%, and a quantity, 0.007%, representing the renewals of renewals in the year of placement of the sum of all the renewals appearing in the column above. This figure is obtained by multiplying the sum of all placements made in the interval, thus, 3.38 + 0.012 + 0.023 = 3.415%, by a factor given by the expression $\frac{r}{2-r}$, where r is the renewal rate (0.0046) of the first interval (0–10) expressed as a decimal. Substituting in this expression gives:

$$3.415 \times \left(\frac{0.0046}{2.00 - 0.0046}\right) = 3.415 \times 0.0023053 = 0.007\%$$

The total renewals for the interval are therefore 3.415 + 0.007 = 3.422%. This figure represents the total renewals made in the interval (10–20), and since the units are new, they will in turn begin to go out of service at the same rate as did the group of original units. All the renewal multipliers are therefore again applied to this figure and the distribution, shown in the horizontal column containing the quantity 0.007 and beginning at the right of the broken line and ending at the left, is obtained. The sum of the figures in this horizontal column will be seen to total 3.422% as shown by the figure in the vertical column on the extreme right of table, as well as at the foot of the vertical column.

Derivation of Expression $\frac{r}{2-r}$.—As illustrated above, the expression $\frac{r}{2-r}$ is used in the calculation of the renewals of units during the same year in which the new units are installed. It is the factor by which the sum of the renewals during any one year is multiplied to give the renewals of those units during the year of placement. The development of this expression is as follows:

Let T = total renewals during any year
r = rate of renewals during first age interval
R = total renewals during year of all renewals during year = $\frac{1}{2}Tr$.[1]
S = sum of all renewals during year except R

Then $T = S + R$
or $R = T - S = \frac{1}{2}Tr$

The ½ appears because only one-half of the first year remains, due to assuming that the replacements are made at middle of interval. From this:

$$T - \tfrac{1}{2}Tr = S$$

or
$$T\left(1 - \frac{1r}{2}\right) = S$$

or
$$T = \frac{S}{1 - \frac{1}{2}r}$$

Subtracting S from each side of equation gives:

$$T - S = \frac{S}{1 - \frac{1}{2}r} - S = R$$

$$R = \frac{S}{1 - \frac{1}{2}r} - \frac{S(1 - \frac{1}{2}r)}{1 - \frac{1}{2}r}$$

$$= \frac{S - S + \frac{1}{2}Sr}{1 - \frac{1}{2}r}$$

$$= \frac{\frac{1}{2}Sr}{1 - \frac{1}{2}r} = \frac{Sr}{2 - r}$$

Therefore:
$$R = S\,\frac{(r)}{(2 - r)}$$

which shows that the renewals of renewals in the year of placement are obtained by multiplying S, the sum of all replacements except R, by a factor $\frac{(r)}{(2 - r)}$ in which r is the

[1] This expression is only approximate. To correctly express this quantity would require the use of the exponential expression of the form $S = Pe^{rt}$. This expression is similar to the compound interest formula used when money is to be compounded an infinite number of times annually.

renewal rate of the first age interval. The value of the expression $\frac{(r)}{(2-r)}$ in every case will be seen to be just slightly larger than $\frac{r}{2}$, which is one-half of the renewal rate of the first age interval. In the case illustrated above,

$$\frac{r}{2} = \frac{0.0046}{2} = 0.0023, \quad \text{and} \quad \frac{0.0046}{2.00 - 0.0046} = 0.0023053$$

Normal Annual Renewals.—If the calculations of the annual renewals are continued over a number of life cycles the renewals will be found to approach a constant value. During the early years of the property life, the annual renewals vary widely from year to year. However, as the property acquires more age these oscillations in renewals gradually dampen until finally the fluctuations disappear entirely and the renewals become constant in value. When the property reaches this stage in its life history it is said to be in its "ultimate" condition; and the constant value of renewals which continues from then on is known as the "normal" annual renewals.

Equation for Normal Annual Renewals.—The equation for the per cent normal annual renewals is of simple form relating the original units and average life thus:

$$\text{Per Cent Normal Annual Renewals} = \frac{\text{Per Cent Original Units (100\%)}}{\text{Average Life}}$$

or if expressed in units instead of per cent,

$$\text{Normal Annual Renewals} = \frac{\text{Number of Original Units}}{\text{Average Life}}$$

With the values of average life available for each of the 52 property groups it becomes a simple matter to determine the per cent normal annual renewals for each group. The per cent normal annual renewals is indicated on each chart by a horizontal line drawn parallel to the lower edge of the chart at a distance equal to the per cent normal annual renewals. It is

Table 51. Per Cent Total Annual Renewals for Seven Graduated Type Mortality and Frequency Curves

Age Interval in Per Cent of Average Life	Types						
	I	II	III	IV	V	VI	VII
0– 10	0.000	0.000	0.000	0.000	0.220	0.260	0.461
10– 20	0.000	0.000	0.040	0.240	0.742	1.013	1.951
20– 30	0.000	0.010	0.199	1.000	1.619	2.153	3.422
30– 40	0.000	0.090	0.680	1.682	2.845	3.532	4.762
40– 50	0.050	0.480	1.730	3.151	4.332	5.033	5.944
50– 60	0.270	1.680	3.703	5.073	5.944	6.495	6.980
60– 70	1.270	4.390	6.429	7.253	7.538	7.825	7.848
70– 80	4.610	8.880	9.647	9.399	8.961	8.945	8.587
80– 90	12.750	14.373	12.659	11.160	10.145	9.805	9.178
90–100	25.960	18.787	14.648	12.328	10.983	10.422	9.659
100–110	34.160	19.748	14.918	12.749	11.465	10.788	10.033
110–120	20.347	16.406	13.893	12.472	11.575	10.930	10.296
120–130	0.622	10.505	11.631	11.659	11.399	10.781	10.465
130–140	0.133	5.325	9.166	10.610	11.021	10.761	10.558
140–150	0.462	3.220	7.395	9.620	10.519	10.524	10.568
150–160	1.376	4.060	6.844	8.947	10.007	10.272	10.522
160–170	3.519	6.412	7.439	8.715	9.594	10.038	10.428
170–180	7.643	9.258	8.708	8.882	9.344	9.847	10.300
180–190	13.816	11.956	10.056	9.054	9.302	9.716	10.150
190–200	20.153	13.810	11.073	9.820	9.484	9.674	9.998
200–210	22.694	14.278	11.540	10.333	9.703	9.692	9.861
210–220	18.345	13.256	11.470	10.585	9.988	9.774	9.760
220–230	9.533	11.196	10.995	10.621	10.194	9.895	9.723
230–240	2.837	8.947	10.325	10.498	10.280	9.991	9.751
240–250	1.241	7.351	9.681	10.290	10.279	10.074	9.881
250–260	4.502	6.910	9.252	10.067	10.233	10.103	9.978
260–270	4.958	7.596	9.126	9.888	10.142	10.100	10.035
270–280	8.650	8.988	9.285	9.783	10.057	10.077	10.061
280–290	13.097	10.503	9.620	9.762	9.988	10.043	10.074
290–300	16.922	11.631	10.002	9.809	9.944	10.012	10.072
300–310	18.261	12.069	10.304	9.895	9.929	9.990	10.057
310–320	16.005	11.770	10.448	9.988	9.935	9.974	10.044
320–330	11.086	10.926	10.429	10.070	9.959	9.966	10.031
330–340	6.133	9.880	10.280	10.112	9.988	9.964	10.014
340–350	3.499	9.008	10.077	10.124	10.016	9.968	10.001
350–360	3.674	8.584	9.886	10.103	10.036	9.978	9.993
360–370	5.789	8.694	9.766	10.065	10.051	9.987	9.991
370–380	8.901	9.231	9.735	10.021	10.052	9.993	9.992
380–390	12.246	9.955	9.788	9.984	10.045	9.998	9.994
390–400	14.880	10.591	9.890	9.967	10.037	10.001	9.993
400–410	15.770	10.946	10.000	9.968	10.025	10.001	10.001
410–420	14.416	10.936	10.082	9.977	10.015	9.999	10.003
420–430	11.314	10.613	10.124	9.994	10.007	9.999	10.009
430–440	7.854	10.132	10.114	10.012	10.007	9.997	10.009
440–450	5.527	9.669	10.072	10.026	10.006	9.994	10.012
450–460	5.130	9.383	10.015	10.034	10.011	9.991	10.011
460–470	6.495	9.337	9.958	10.032	10.018	9.990	10.008
470–480	8.906	9.522	9.923	10.028	10.019	9.990	10.009
480–490	11.497	9.842	9.915	10.024	10.023	9.991	10.007
490–500	13.465	10.171	9.932	10.014	10.024	9.991	10.006
500–510	14.161	10.401	9.962	10.007			
510–520	13.309	10.465	9.995	10.005			
520–530	11.241	10.365	10.016	10.005			
530–540	8.796	10.154	10.030	10.005			
540–550	6.963	9.922	10.026	10.010			
550–560	6.401	9.748	10.009	10.013			
560–570	7.203	9.682	9.995	10.017			
570–580	8.932	9.729	9.980	10.016			
580–590	10.907	9.861	9.969	10.017			
590–600	12.439	10.025	9.969	10.016			

this line about which the annual renewals oscillate until they become constant. (See Figure 81.) When the annual renewals have become constant in value the annual renewals curve and the normal annual renewals line coincide. From the foregoing it thus becomes apparent that the axis of oscillation of the total annual renewals curve can be predicted and established without the calculation of the total annual renewals curve itself. Furthermore, by noting the magnitude of the oscillations above and below this line some notion of the maximum and minimum annual renewals can be obtained.

Seven Type Annual Renewal Curves.—To show fully the manner in which the annual renewals vary with time, the annual renewals have been calculated for each of the seven type curves developed in Chapters 3 and 4. The illustration used in the foregoing paragraphs in explaining the procedure employed in making the renewal calculations showed the calculations for Type VII of the type curves. Table 51 shows the calculated values of annual renewals for each of the seven type curves for property ages from 0 to 500% and 600% of average life. To show better the manner of variation these values of annual renewals have been plotted in Figure 82. It will be noted that the calculations must extend over a number of life cycles to show the law of renewals adequately. This figure strikingly shows the oscillatory character of the annual renewals, and also distinctly shows how these oscillations dampen with time, until finally all oscillations cease and the renewals become constant. When this condition obtains, the property has reached its ultimate condition and the annual renewals become a definite fixed quantity equal to the quotient of 100% and average life. It should also be pointed out that since the seven type mortality tables on which these renewal curves are based cover the range of normal mortality tables to be found in the realm of physical property, the seven renewal curves indicate the range and spread of this class of curves.

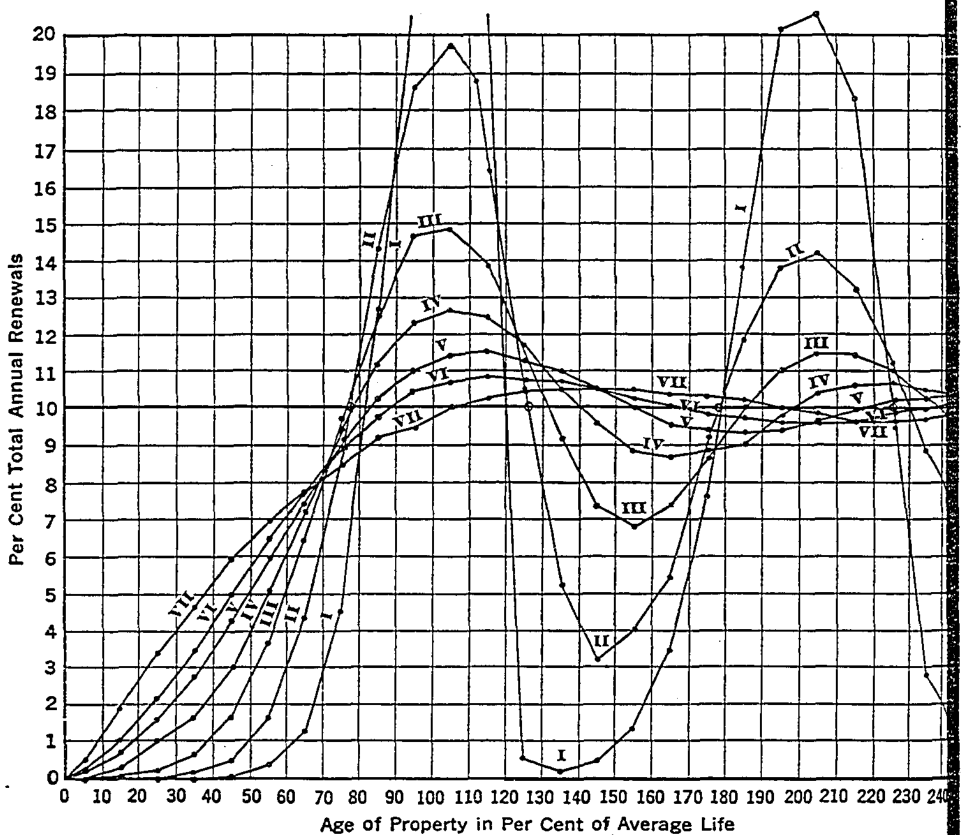
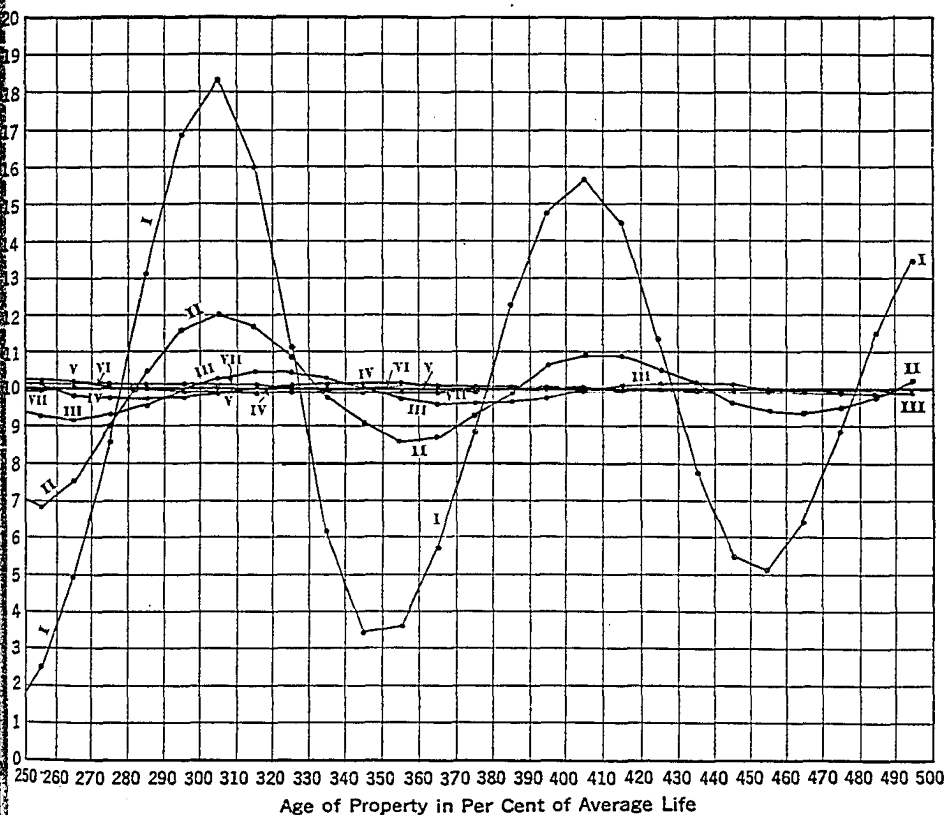


Figure 82. Oscillatory Annual Renewal Cu



Seven Graduated Mortality and Frequency Types

Equations of Oscillatory Annual Renewal Curves.—As already noted, the seven annual renewal curves appear to be of the damped oscillatory type. Such curves can be represented by an equation of the form $y = de^{-ax} \sin bx$. This equation has been successfully applied to these curves and the following expressions have been obtained for the seven type curves:

Type Number	
I	$Y = 10 + 21.7\, e^{-0.00284x} \cdot \sin(0.0633x + 1.0932)$
II	$Y = 10 + 20.0\, e^{-0.00731x} \cdot \sin(0.0608x + 1.5932)$
III	$Y = 10 + 17.06\, e^{-0.01122x} \cdot \sin(0.0574x + 1.923)$
IV	$Y = 10 + 12.3\, e^{-0.01332x} . \sin(0.0524x + 2.1707)$
V	$Y = 10 + 7.57\, e^{-0.01378x} \cdot \sin(0.0482x + 2.243)$
VI	$Y = 10 + 4.60\, e^{-0.01375x} \cdot \sin(0.040x + 2.763)$
VII	$Y = 10 + 3.81\, e^{-0.01312x} \cdot \sin(0.0405x + 2.0282)$

These equations consist essentially of two parts—namely, the periodic function "sin bx" which causes the values of y to oscillate, and "e^{-ax}," called the dampening factor, which causes the oscillations gradually to decrease in amplitude.

These equations do not satisfactorily fit the curves until after the first crossing of the $y = 10$ axis. This is due to the fact that the curve starts at $y = 0$, but oscillates about the axis, $y = 10$. A true oscillatory curve would start from $y = 10$ in this case.

Method of Obtaining Constants in Equations.—The procedure of obtaining the values of the constants in the equation $y = de^{-ax} \sin bx$ consists essentially of five distinct steps. These steps will be discussed relative to type curve III which will be used as an illustration.

1. Since the curve is oscillatory about the $y = 10$ axis, 10 may be subtracted from all y values, making the y values conform to the normal oscillatory curve, thus $y - 10 = de^{-ax} \sin bx$. The value 10 is added to the final equation to correct for this procedure.

2. The sine function of the equation may be calculated

directly from the zero values of the wave. Between the values of $x = 76$ and $x = 405$ (Type III, Figure 82) the curve passes through three complete oscillations.

This distance is equivalent to $2\pi \times 3 = 6\pi$ radians. Each division on the x scale is thus equal to $\frac{6\pi}{(405 - 76)}$ radians, or 0.0574 radians.

Substituting this value in the sine function gives sine $(0.0574x + \theta)$. The angle θ must be introduced to make the sine of the whole angle equal to zero for those values of x where the curve crosses the axis. Since the first few alternations are the most important, this adjustment is made for the value of $x = 76$. Substituting $x = 76$ gives sine $(4.36 + \theta)$, making it necessary to add $\theta = 1.923$ in order to make the angle equal to 2π. The sine function therefore is:

$$\text{sine } (0.0574x + 1.923)$$

3. In a dampened wave-train, the ratio of the maximum amplitudes of any two successive cycles is constant. If A_1 and A_2 represent the values of the amplitudes of two successive cycles, the ratio of $\frac{A_1}{A_2} = \frac{1}{e^{-az}}$, where z is the period between cycles. If then values of A_1 and A_2 and z are obtained from the plotted curve the constant, a, can be evaluated. For

Table 52. Amplitudes of Maximum Values of Renewal Oscillations

	A_1	A_2	Ratio of $\frac{A_1}{A_2}$	Value of z at Maximum
ABOVE	49.0 15.2 4.5	15.2 4.5 1.3	3.221 3.38 3.40	105 205 320
BELOW	31.5 8.9 2.8	8.9 2.8 0.82	3.54 3.18 3.41	155 265 375
			Average Ratio =3.35	Average Difference =108.75

Type III the amplitudes of successive maximum values above and below the axis of oscillation and the values of z are as shown in Table 52. By substituting the average values found for the ratio of maximum amplitudes and period, the constant a is found, thus,

$$3.35 = e^{a \times 108.75}$$

or $$\log 3.35 = a \times 108.75 \times \log e$$

whence $$0.52504 = a \times 108.75 \times 0.4343$$

$$a = 0.01122$$

4. To find the value of the last constant, d, it is only necessary to substitute a series of values from the original data and thereby obtain a number of values from which an average can be obtained. To illustrate, let

$$x = 103.5, \text{ for which } y = 14.88$$
$$14.88 = 10 + de^{-103.5 \times 0.01122} \sin(5.94 + 1.923)$$
$$14.88 = 10 + de^{-1.162} \sin 7.863 \text{ radians}$$
$$4.88 = de^{-1.162} \times 1$$
$$\text{or } \log 4.88 = \log d - 1.162 \log e$$
$$\log d = \log 4.88 + 1.162 \log e$$
$$= 0.68842 + 1.162 \times 0.4343$$
$$= 1.19242$$
$$\text{and } d = 15.58$$

By taking other values, such as $x = 156$, $y = 7.0$, d is found to equal 18.4. Averaging a number of such values resulted in $d = 17.06$.

5. The last step consists in adding 10 to the equation as found. This corrects for the procedure in the first step.

Comparison of Equation Constants.—In order to show the change in value of the constants from Types I to VII, the values of constants a and d are plotted in Figure 83. The constant d decreases in going from Types I to VII, while constant a increases in value. This is as expected since constant d determines the maximum amplitude, whereas constant a determines the rapidity of dampening. Constant d varies from

21.7 for Type I to 3.81 for Type VII; and constant *a* varies from 0.00284 for Type I to 0.01312 for Type VII.

Significance of Annual Renewal Equations.—The fact that the seven annual renewal curves are susceptible to mathe-

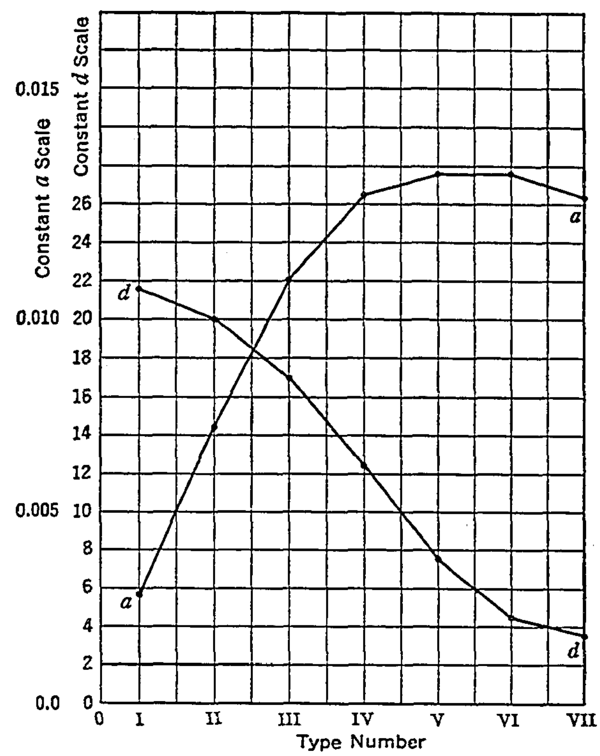


Figure 83. Variation of Constants in Oscillatory Annual Renewal Equations

matical representation is of great significance; indeed, the fact that the simple equation for damped oscillations can be made to fit these curves classes these curves with other natural phenomena. The phenomenon of damped vibrations or oscillations is fundamental in the physical realm, and any phenomenon showing such behavior is thus worthy of note. These equations, therefore, are but further evidence in indicating the similarity between the mortality laws of physical property and the many laws in the field of physical science.

Time Required for Annual Renewals to Become Constant.—From the previous paragraphs it was seen that the total annual renewal curves are oscillatory curves, the oscillations being above and below the normal annual renewal line. As time advances the oscillations become smaller and smaller, until finally the oscillations are damped out completely and the curve becomes a straight horizontal line. In order to study the time required for the annual renewals to become constant some practical basis has to be set for this determination. The basis

Table 53. Relation of Time for Annual Renewals to Become Constant and Slope of Mortality Curve for 52 Property Groups

Mortality Group Number	Ratio of Time Required for Annual Renewals to Become Constant to Within 5% of Normal to Average Life	Per Cent of Average Life Between ¾ and ¼ Quartiles	Mortality Group Number	Ratio of Time Required for Annual Renewals to Become Constant to Within 5% of Normal to Average Life	Per Cent of Average Life Between ¾ and ¼ Quartiles
1	1.65	85	28	2.20	69
2	1.48	70	29	5.28	27
3	1.36	60			
4	1.74	37	30	3.71	47
5	3.51	47	31	1.32	54
6	2.09	70	32	2.25	70
7	6.67	21	33	1.62	43
8	2.16	76	34	3.10	32
9	1.83	95	35	4.95	29
			36	6.15	27
10	2.44	80	37	4.02	31
11	2.17	56	38	3.78	32
12	2.92	74	39	6.36	23
13	2.28	76			
14	2.04	53	40	2.77	39
15	1.68	48	41	14.00	17
16	1.42	51	42	14.60	14
17	2.20	69	43	8.07	26
18	2.37	82	44	7.30	24
19	2.41	102	45	4.58	30
			46	5.70	28
20	1.44	52	47	6.28	25
21	1.22	63	48	4.40	25
22	1.97	85	49	5.76	27
23	1.63	50			
24	1.97	51	50	4.26	32
25	1.01	62	51	2.17	35
26	2.18	53	52	7.10	26
27	2.00	50			

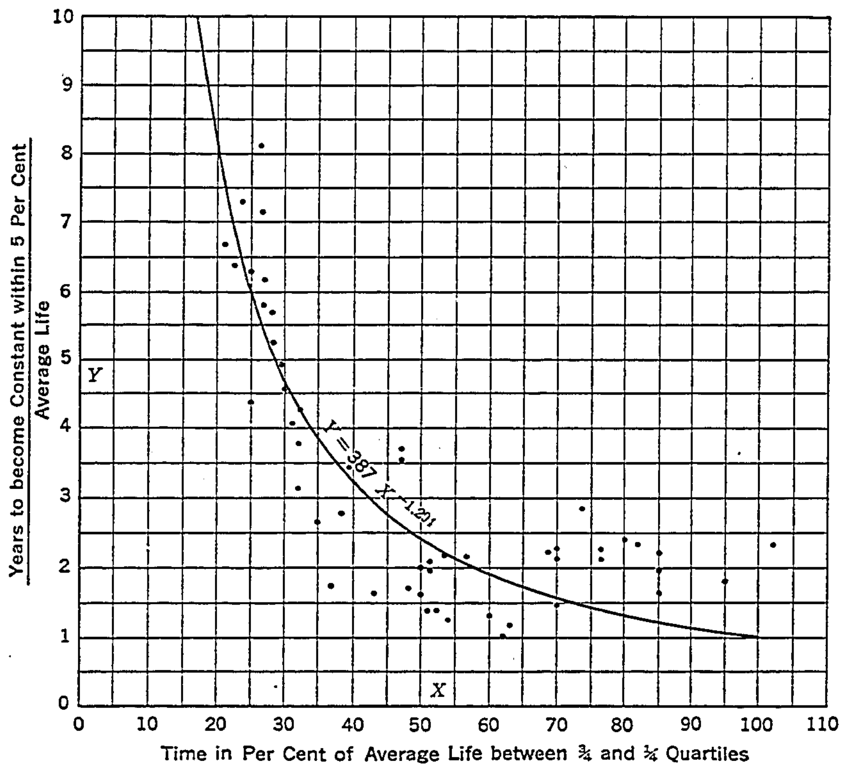


Figure 84. · Relation Between Time Required for Annual Renewals to Become Constant Within 5% and Time Between 3/4 and 1/4 Quartiles

used was to observe the time required for the renewals to become constant within 5% of normal value.

It had become apparent in making the annual renewal calculations that the shape of the curve had considerable bearing upon the number of cycles the renewals would continue to oscillate. In the development of type mortality curves in Chapter 3, it was found that the shape of a mortality curve could be best indicated by the time between the 3/4 and 1/4 quartiles expressed in per cent of average life. Since the time required for the annual renewals to become constant also seems to depend on the shape of the mortality curve, this quantity (time between 3/4 and 1/4 quartiles) should also be a good criterion for it.

In Table 53 are tabulated the years required for the renewals to become constant to within 5%. The years are expressed as ratios of average life. This table also shows the corresponding values of the time in per cent of average life between the 3/4 and 1/4 quartiles for each of the 52 property groups. On Figure 84 this function is plotted against the number of average life cycles required for the renewals to become constant within 5% of normal value. The figures indicate that the function is quite satisfactory as an index of the length of time that the renewals continue to oscillate. The points fall along a well-outlined hyperbolic curve the equation for which is given below. Thus, again this function proved to be a good criterion.

This hyperbolic curve clearly shows that as the mortality curves become steeper, the annual renewals oscillate more violently and therefore require a longer time to dampen out. This is reasonable for, as the main part of the mortality curve becomes more nearly vertical, the replacements of the original units are crowded into less and less years. This reduction in time can only be made up by a greater number of replacements per year.

Method of Obtaining Equation.—Points obtained from an approximate curve were plotted on log-log paper and a straight line was evident. A line was drawn through these points by sight. Although this line did not fit all the points it probably represents the law or relation quite well. The form of the equation, therefore, is as follows:

$$Y = ax^b$$

which can be written

$$\log Y = \log a + b \log x$$

Two pairs of points from the straight line were selected thus,

$$Y_1 = 6 \qquad x_1 = 25$$
$$Y_2 = 1 \qquad x_2 = 100$$

The logarithms of these numbers are:

$$\log 6 = 0.7782 \qquad \log\ 25 = 1.3979$$
$$\log 1 = 0.0000 \qquad \log 100 = 2.0000$$

Substituting in the above form and solving simultaneously,

$$0.7782 = \log a + b \times 1.3979$$
$$0.0000 = \log a + b \times 2.0000$$
$$0.7782 = \quad 0 - b \times 0.6021$$
$$b = \frac{-\ 0.7782}{0.6021} = -\ 1.294$$

Substituting the value of b and solving for a,

$$0.0000 = \log a - 1.294 \times 2.0000$$
$$-\log a = -\ 2.588$$
$$a = 387$$
$$y = 387 \times X^{-1.294}$$

Relation of Maximum Annual Renewals to Normal Annual Renewals.—In order that adequate provision may be made for the replacement of units as they go out of service, it is necessary to know the magnitude of the maximum renewals, and the year during which they must be made. The relation of the maximum annual renewals to the normal annual renewals is discussed below, and the time when these renewals can be expected to occur is discussed in the succeeding paragraph.

In Table 54 are given the values of normal annual renewals, maximum annual renewals, and the ratios of these two quantities for each of the 52 property groups. Table 55 shows the frequency with which these ratio values occur in the various intervals.

The results of these tabulations show that the average value of the ratio of maximum annual renewals to normal annual renewals for the 52 cases is

$$\text{Average Ratio of } \frac{\text{Maximum Annual Renewals}}{\text{Normal Annual Renewals}} = 1.647$$

Table 54. Normal and Maximum Annual Renewals, and Time Between ¾ and ¼ Quartiles for 52 Property Groups

Mortality Group Number	Normal Annual Renewals	Maximum Annual Renewals	Ratio of Maximum to Normal Annual Renewals	Time between ¾ and ¼ Quartiles in % of Average Life
1	5.88	7.09	1.21	85
2	7.42	8.11	1.09	70
3	4.69	5.11	1.09	60
4	3.18	6.50	2.05	37
5	6.75	9.55	1.41	47
6	18.97	25.02	1.32	70
7	8.64	22.99	2.66	21
8	7.98	8.64	1.08	76
9	11.47	12.32	1.07	95
10	9.41	9.99	1.06	80
11	10.33	11.60	1.12	56
12	10.79	11.96	1.11	74
13	8.77	10.18	1.16	76
14	7.05	8.61	1.22	53
15	6.73	8.93	1.33	48
16	6.15	6.46	1.05	51
17	8.16	9.50	1.17	69
18	8.16	9.15	1.12	82
19	8.61	10.67	1.24	102
20	12.00	16.52	1.38	52
21	10.16	11.39	1.12	63
22	10.34	11.26	1.09	85
23	10.87	12.74	1.17	50
24	9.36	11.33	1.21	51
25	10.07	10.43	1.04	62
26	8.72	10.48	1.20	53
27	12.54	15.11	1.20	50
28	10.00	11.02	1.10	69
29	8.13	20.10	2.48	27
30	9.51	13.72	1.45	47
31	3.29	3.78	1.15	54
32	4.24	4.74	1.12	70
33	3.94	5.27	1.34	43
34	3.02	5.28	1.75	32
35	5.01	10.43	2.08	29
36	4.80	11.25	2.34	27
37	5.36	10.33	1.93	31
38	5.18	10.57	2.04	32
39	8.95	27.20	3.04	23
40	11.08	15.64	1.41	39
41	7.48	19.12	2.56	17
42	5.82	26.00	4.47	14
43	10.74	25.00	2.32	26
44	9.13	23.00	2.52	24
45	9.56	17.50	1.83	30
46	8.90	17.50	1.97	28
47	9.81	21.00	2.14	25
48	10.22	23.00	2.25	25
49	8.74	18.00	2.06	27
50	9.26	24.50	2.64	32
51	12.06	18.00	1.49	35
52	12.21	27.00	2.20	26
		Total =	85.65	
		Average Ratio =	1.647	

The distribution diagram, Figure 85, however, indicates that 32 of the 52 cases have values between 1.0 and 1.5, eleven of which have a ratio of about 1.15. There are a number of cases where the ratio lies between 2 and 3. These cases raise the value of the average ratio.

It is apparent that the shape of the mortality curve controls the magnitude of the maximum annual renewals. If the curve is very steep a great many units go out on the steep part of the curve, and thus the renewals will be large. On the other hand if the curve is not steep the replacements are not large during any one year and consequently the ratio of maximum to normal is low. From previous studies it has become evident that the time elapsing between the 3/4 and 1/4 quartiles gives a good indication of the shape of a mortality curve and since the magnitude of the maximum renewals apparently also depends upon the shape of the mortality curve, this function was again employed. Table 54 also gives the values of the time between the 3/4 and 1/4 quartiles in per cent of average life, and Figure 86 shows these values plotted against the ratios of maximum to normal annual renewals. Since there appears to be a marked relation, it was decided to use this function and to determine the mathematical expression for the relation.

From an inspection of the curve obtained it appears that the relation is a hyperbolic one. This was tested by drawing an approximate curve through the several points and plotting points from this curve on log-log paper. The equation for this line is:

$$Y \cdot X^{0.677} = 21.06$$

where Y equals the ratio of maximum annual renewals to normal annual renewals, and X represents the time between the 3/4 and 1/4 quartiles in per cent of average life. This equation was then plotted on Figure 86, and the agreement

Table 55. Distribution of Ratios of Maximum to Normal Annual Renewals

Interval	Number of Cases of $\frac{\text{Maximum Annual Renewals}}{\text{Normal Annual Renewals}}$ in Interval
1.0–1.1	9
1.1–1.2	11
1.2–1.3	4
1.3–1.4	4
1.4–1.5	4
1.5–1.6	0
1.6–1.7	0
1.7–1.8	1
1.8–1.9	1
1.9–2.0	2
2.0–2.1	4
2.1–2.2	2
2.2–2.3	1
2.3–2.4	2
2.4–2.5	1
2.5–2.6	2
2.6–2.7	2
2.7–2.8	0
2.8–2.9	0
Above	2
	52

appears to be satisfactory. It may be of interest to note that the exponent is practically 2/3.

Method of Obtaining Equation.—Since the data took the form of a straight line when plotted on log-log paper, it was evident that the form of the equation was,

$$Y = aX^b$$

which can be written,

$$\log Y = \log a + b \log x$$

Taking two pairs of values from the line on log-log paper,

$$Y_1 = 3.36 \qquad X_1 = 15$$
$$Y_2 = 1.00 \qquad X_2 = 90$$

and obtaining the logarithms for same,

$$\log Y_1 = 0.5263 \qquad \log X_1 = 1.1761$$
$$\log Y_2 = 0.0000 \qquad \log X_2 = 1.9542$$

Then substituting in the form above and solving simultaneously,

$$0.5263 = \log a + b \times 1.1761$$
$$\underline{0.0000 = \log a + b \times 1.9542}$$
$$0.5263 = \quad 0 \quad - b \times 0.7781$$
$$b = -\frac{0.5263}{0.7781} = -0.677$$

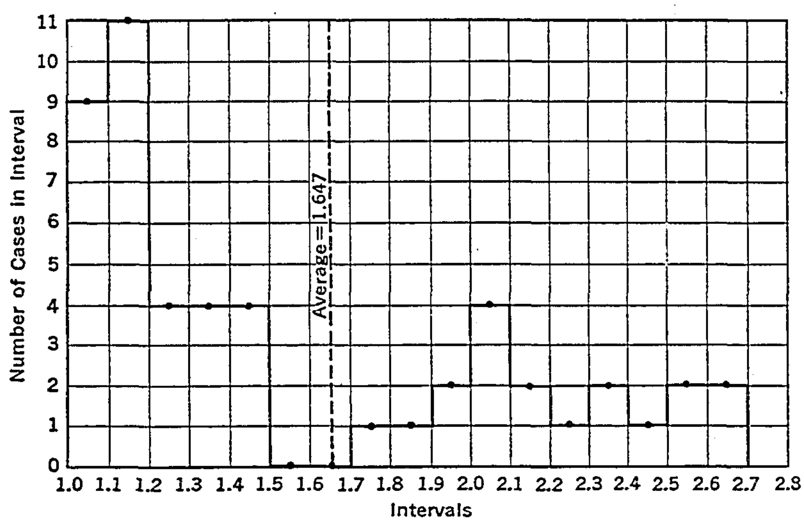


Figure 85. Frequency Diagram of Ratios of Maximum to Normal Annual Renewals

Substituting back and solving for a,

$$0.0000 = \log a - 0.677 \times 1.9542$$
$$-\log a = -1.323$$
$$a = 21.06$$
$$Y = 21.06 \times X^{-0.677}$$

Time of Maximum Annual Renewals.—As stated before, it is not only important to know how large the maximum renewals are going to be, but it is also important to know when the renewals are going to be a maximum. In looking over the original data it appeared at once that there was much similarity between the time of maximum renewals and the modal year. This is quite reasonable because the modal

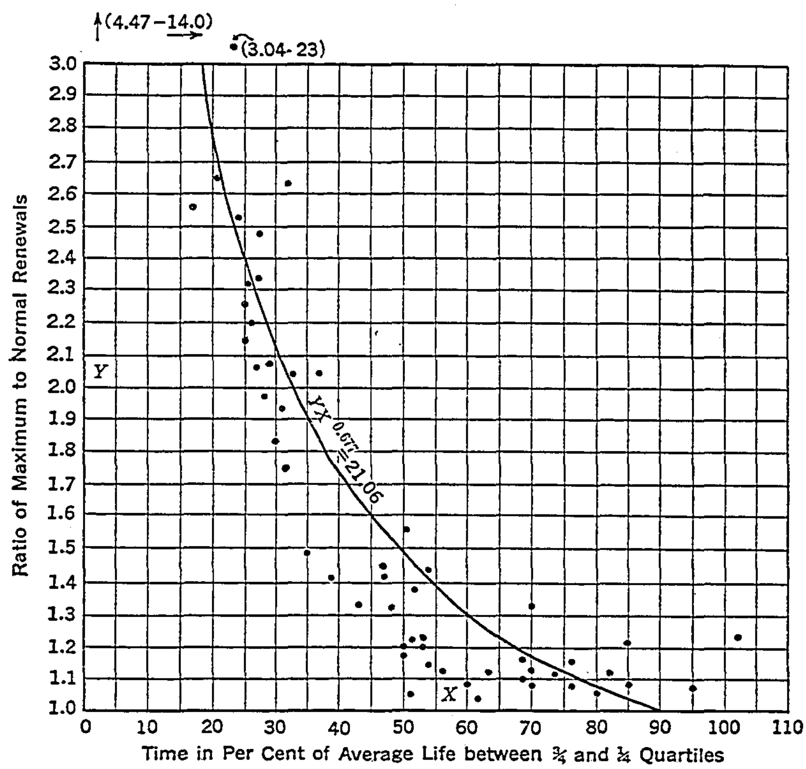


Figure 86. Relation Between Ratio of Maximum to Normal Annual Renewals and Time Between 3/4 and 1/4 Quartiles

year is the year in which the largest number of units of the original group are taken out of service. If the total maximum annual renewals did not include renewals of renewals, the modal year and year of maximum renewals would agree exactly for all cases. But total maximum annual renewals include both renewals of original units and renewals of renewals, and therefore agreement is not possible in all cases, as will be pointed out below. In Table 56 are tabulated the year in which the maximum annual renewals occur, and the modal year for the distribution of original units for each of the 52 property groups. These two quantities are plotted against each other in Figure 87. It will be noted that 25 points lie on the 45° line, indicating that in almost half of the 52 cases the mode is the year in which the total annual renewals are at a maximum. The equation for these points would be,

Year of Maximum Annual Renewals = Modal Year

Only 2 of the remaining cases lie below this line and the remainder lie above. For any point above the line the time of maximum renewals occurs beyond the year of the mode. Only 9 of the cases above the line depart enough from the curve to deserve special attention. These 9 cases are mortality groups No. 1, 3, 8, 10, 18, 22, 25, 32, and 41. These tables, in general, show so little difference between the magnitude in the renewals occurring at the modal year and the actual maximum renewals that no serious error would be made in using the renewals at the mode. Any error made would always be on the safe side as the mode precedes the time of actual maximum renewals. In these tables the renewals of the original units occur so uniformly as to have no perceptible peak, and therefore the mode is not well defined. The renewals of renewals determine the maximum in these cases and therefore carry it beyond the modal year. But this maximum is approached so gradually and is in many cases only slightly above the value of the renewals at the mode that no serious error would be made in assuming the maximum to coincide with the mode. The relation $Y = X$ can therefore be safely taken as the expression of the time of maximum renewals.

Relation of Minimum Annual Renewals to Normal Annual Renewals.—Since it was found that the time between

Table 56. Year of Maximum Annual Renewals, and Modal Year for 52 Property Groups

Mortality Group Number	Year of Maximum Annual Renewals	Modal Year	Mortality Group Number	Year of Maximum Annual Renewals	Modal Year
1	34	10	27	8	7
2	10	10	28	14	10
3	22	16	29	14	14
4	27	27			
5	21	16	30	14	13
6	7	7	31	40	30
7	13	13	32	43	17
8	21	8	33	21	21
9	7	6	34	30	30
			35	21	21
10	22	7	36	21	21
11	10	10	37	21	21
12	10	13	38	22	22
13	20	14	39	13	13
14	17	16			
15	14	14	40	11	11
16	11	16	41	25	15
17	16	15	42	19	19
18	25	8	43	10	10
19	8	7	44	12	12
			45	12	12
20	8	7	46	13	13
21	11	10	47	11	11
22	18	5	48	10	10
23	10	7	49	14	14
24	13	11			
25	12	9	50	10	10
26	10	10	51	9	9
			52	9	9

the 3/4 and 1/4 quartiles expressed in per cent of average life was a satisfactory function to use in expressing the relation between the ratio of maximum annual renewals to normal annual renewals, it was assumed that it would also be a satisfactory function for expressing the relation between minimum annual renewals and normal annual renewals. Table 57 gives the values for minimum annual renewals, normal annual renewals, ratio of minimum to normal renewals, and time between 3/4 and 1/4 quartiles expressed in per cent of average life. In Figure 88 the ratios between minimum and normal annual renewals are plotted against the time between the 3/4 and 1/4 quartiles. It will be noted that the points line up well and present a well-defined tendency. Values taken from a rough curve drawn through these points, however, would not form a straight line on log-log paper; but by subtracting the ordinates of this curve from 1.0 the hyperbolic curve form is obtained. The equation for this curve is:

$$Y = 448 \times X^{-2.03}$$

and the equation for the former curve is

$$Y = 1.0 - 448 \times X^{-2.03}$$

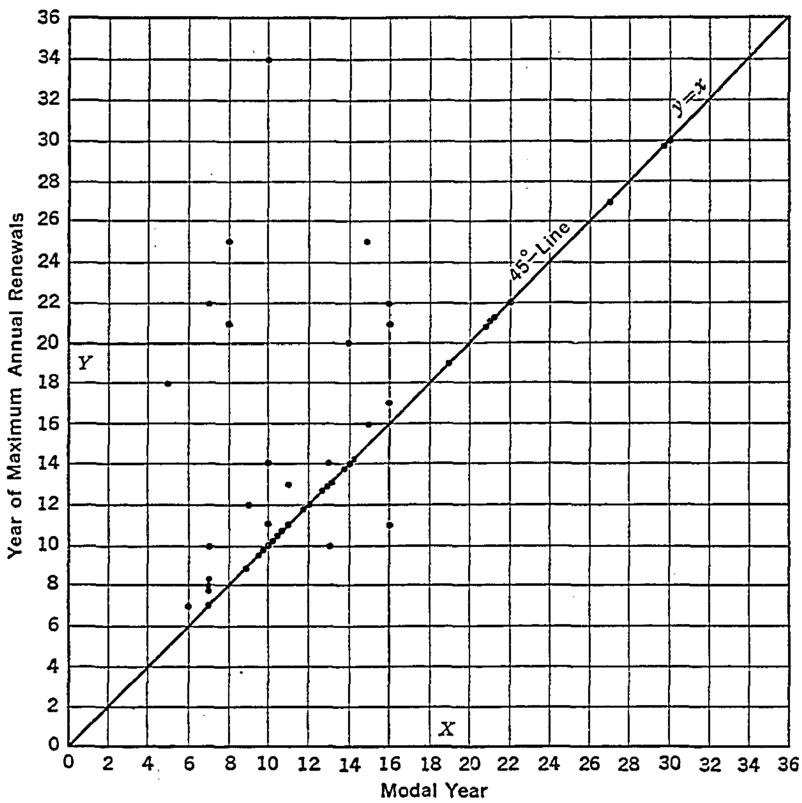


Figure 87. Relation Between Year of Maximum Annual Renewals and Modal Year for 52 Property Groups

The manner of obtaining this equation is given in the following paragraph. Points were then computed with this equation and plotted on Figure 88, giving the curve shown. It will be noted that the agreement is very good. It is of interest to note also the value of the exponent as being practically 2.0.

Table 57. Minimum and Normal Annual Renewals for 52 Property Groups

Mortality Group Number	Minimum Annual Renewals	Normal Annual Renewals	Ratio of Minimum Annual Renewals / Normal Annual Renewals	Per Cent of Average Life Between ¾ and ¼ Quartiles
1	5.15	5.88	0.88	85
2	6.56	7.42	.89	70
3	4.30	4.69	.92	60
4	2.57	3.18	.81	37
5	4.58	6.75	.68	47
6	17.26	18.97	.91	70
7	2.83	8.64	.33	21
8	7.44	7.98	.93	76
9	9.38	11.47	.82	95
10	8.95	9.41	.95	80
11	9.76	10.33	.94	56
12	10.60	10.79	.98	74
13	7.31	8.77	.83	76
14	6.49	7.05	.92	53
15	5.60	6.73	.83	48
16	6.05	6.15	.98	51
17	7.04	8.16	.86	69
18	7.44	8.16	.91	82
19	7.06	8.61	.82	102
20	9.48	12.00	.79	52
21	9.89	10.16	.97	63
22	9.89	10.34	.95	85
23	9.70	10.87	.89	50
24	8.23	9.36	.88	51
25	9.87	10.07	.98	62
26	7.91	8.72	.91	53
27	10.45	12.54	.83	50
28	9.03	10.00	.90	69
29	2.48	8.13	.31	27
30	6.92	9.51	.73	47
31	3.27	3.29	.99	54
32	3.75	4.24	.88	70
33	3.64	3.94	.92	43
34	1.95	3.02	.65	32
35	2.03	5.01	.41	29
36	2.20	4.80	.46	27
37	2.75	5.36	.51	31
38	3.00	5.18	.58	32
39	1.81	8.95	.20	23
40	8.25	11.08	.74	39
41	0.057	7.48	.00	17
42	0.00	5.82	.00	14
43	1.98	10.74	.18	26
44	2.82	9.13	.31	24
45	2.76	9.56	.29	30
46	0.68	8.90	.07	28
47	2.27	9.81	.23	25
48	4.85	10.22	.47	25
49	0.48	8.74	.06	27
50	4.00	9.26	.43	32
51	8.69	12.06	.72	35
52	0.20	12.24	.02	26

Method of Obtaining Equation for Curve.—When the ordinates, subtracted from 1.0, plotted on log-log paper took the form of a straight line, it was evident that the equation was of the form:

$$Y = a \cdot x^b$$

which can be written

$$\log Y = \log a + b \log x.$$

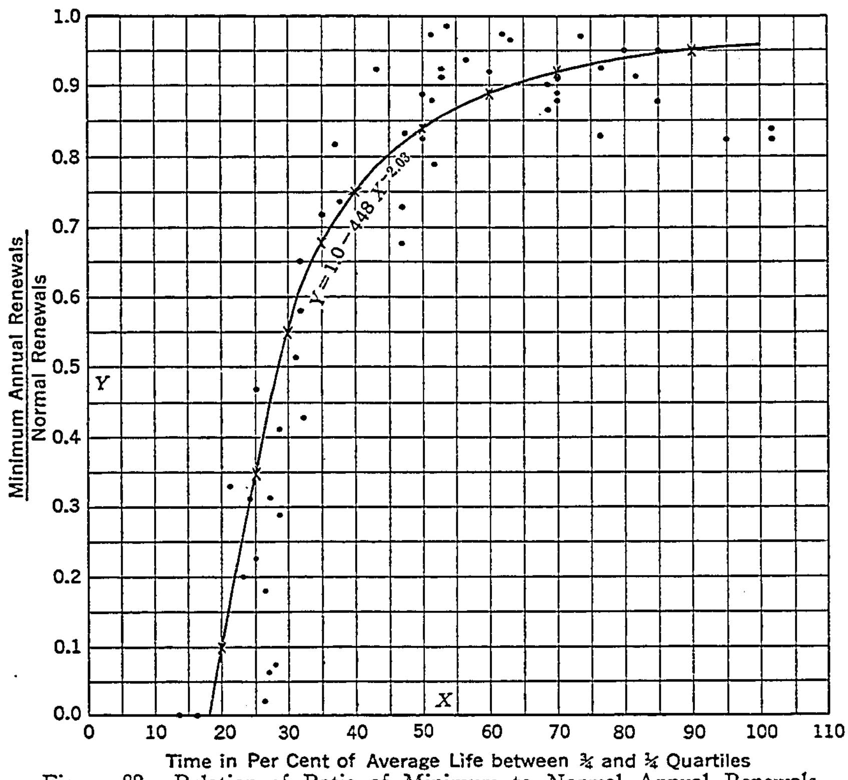


Figure 88. Relation of Ratio of Minimum to Normal Annual Renewals and Time Between 3/4 and 1/4 Quartiles

Then taking two pairs of values from the line on log-log paper,

$$Y_1 = 0.65 \qquad X_1 = 25$$
$$Y_2 = 0.11 \qquad X_2 = 60$$

and obtaining the logarithms for same,

$$\log 0.65 = 9.8129 - 10 \qquad \log 25 = 1.3979$$
$$\log 0.11 = 9.0414 - 10 \qquad \log 60 = 1.7782$$

Then substituting in the form above and solving simultaneously,

$$\begin{aligned} 9.8129 - 10 &= \log a + b \times 1.3979 \\ 9.0414 - 10 &= \log a + b \times 1.7782 \\ \hline 0.7715 &= 0 \quad - b \times 0.3803 \end{aligned}$$

$$b = -\frac{0.7715}{0.3803} = -2.03$$

Now substituting for b and solving for a,

$$\begin{aligned} 9.0414 - 10 &= \log a - 2.03 \times 1.7782 \\ -\log a &= -2.6514 \\ a &= 448 \\ Y &= 448\, X^{-2.03} \\ Y &= 1.0 - 448\, X^{-2.03} \end{aligned}$$

Relation of Time of Minimum Annual Renewals to Time of Maximum Annual Renewals.—This relation was investigated by obtaining the ratios of the time of minimum annual renewals to time of maximum annual renewals for the 52 property groups, computing the average of the 52 ratios, and then studying the distribution of these ratios. Tables 58 and 59 give the values of these ratios and their distribution over the range. An analysis of the data in these tables shows that the average value of the ratio between

$$\frac{\text{Time of Minimum Annual Renewals}}{\text{Time of Maximum Annual Renewals}} = 1.478$$

The distribution curve, Figure 89, however, shows that the greatest number of cases have a ratio of about 1.35. Fifteen cases have a ratio of about 1.35. Thirty of the 52 cases lie between 1.2 and 1.5. The 41 cases which actually make up the major part of the curve lie very close together, their range

Table 58. Years of Minimum and Maximum Annual Renewals for 52 Property Groups

Mortality Group Number	Year of Minimum Annual Renewals	Year of Maximum Annual Renewals	Ratio of Year of Minimum Annual Renewals / Maximum Annual Renewals
1	36	34	1.06
2	18	10	1.80
3	29	22	1.32
4	41	27	1.52
5	25	21	1.19
6	9	7	1.28
7	17	13	1.31
8	28	21	1.33
9	9	7	1.28
10	26	22	1.18
11	21	10	2.10
12	16	10	1.60
13	23	20	1.15
14	29	17	1.71
15	21	14	1.50
16	20	11	1.82
17	24	16	1.50
18	28	25	1.12
19	11	8	1.38
20	10	8	1.25
21	20	11	1.82
22	22	18	1.22
23	14	10	1.40
24	20	13	1.54
25	26	12	2.16
26	24	10	2.40
27	15	8	1.88
28	21	14	1.50
29	19	14	1.36
30	19	14	1.36
31	95	30	3.17
32	49	43	1.14
33	38	21	1.81
34	52	30	1.73
35	31	21	1.48
36	31	21	1.48
37	29	21	1.38
38	29	22	1.32
39	16	13	1.23
40	14	11	1.27
41	31	25	1.24
42	21	19	1.10
43	14	10	1.40
44	17	12	1.42
45	16	12	1.33
46	16	13	1.23
47	15	11	1.36
48	15	10	1.50
49	16	14	1.14
50	14	10	1.40
51	12	9	1.33
52	12	9	1.33
		Total =	76.83
		Average Ratio =	1.478

being from 1.0 to 1.6 or an average of about 1.3. The average ratio of 1.478 was raised by 7 cases lying between 1.7 and 1.9 and several cases having a ratio above 2.0.

Maximum and Minimum Annual Renewals for Seven Type Curves.—It is of interest to note the magnitude of the maximum and minimum annual renewals for the seven type curves. These values are tabulated in Table 60 and plotted on Figure 90. Since the normal annual renewals equal 10 in each case, the number of times by which the maximum exceeds normal value can be easily observed. The same is true of the minimum value of annual renewals. The maximum annual renewals decrease with increase in type number, whereas the minimum annual renewals increase with increase in type number. The maximum annual renewals decrease from almost 3.5 times normal to nearly normal in passing from

Table 59. Distribution of Cases of Ratio of Time of Minimum Annual Renewals to Time of Maximum Annual Renewals by Intervals

Interval	Number of Cases of Ratio of $\frac{\text{Time of Minimum Annual Renewals}}{\text{Time of Maximum Annual Renewals}}$
1.0–1.1	2
1.1–1.2	7
1.2–1.3	8
1.3–1.4	15
1.4–1.5	7
1.5–1.6	2
1.6–1.7	0
1.7–1.8	3
1.8–1.9	4
1.9–2.0	0
2.0–2.1	1
2.1–2.2	1
2.2–2.3	0
2.3–2.4	1
	52

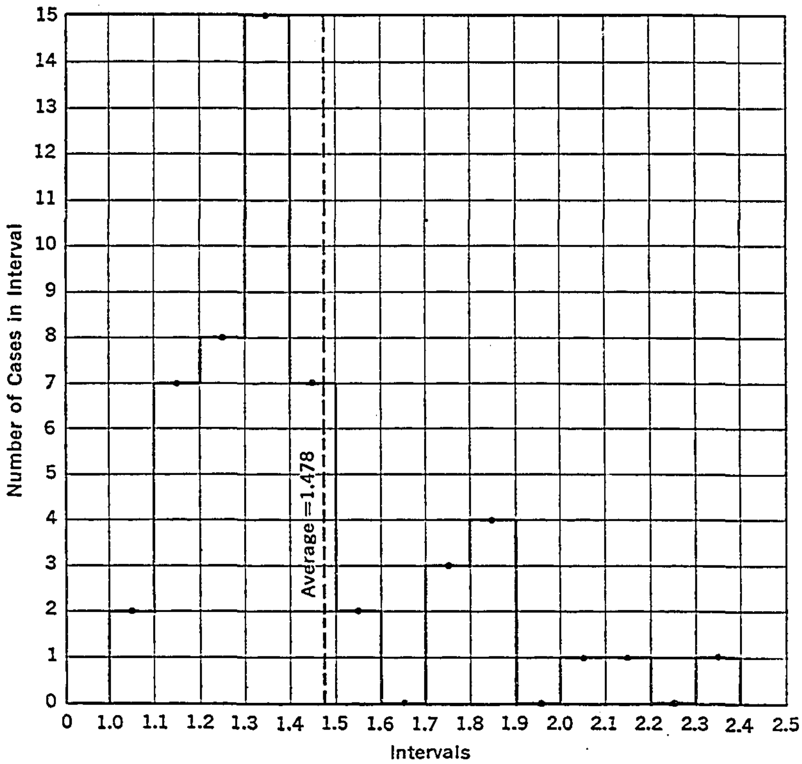


Figure 89. Frequency Diagram of Ratio of Year of Minimum to Maximum Annual Renewals

Types I to VII, while the minimum annual renewals increase from practically zero to normal in passing over the same range of type numbers. Both maximum and minimum annual renewals approach the normal annual renewal line as the type numbers increase; in fact, the normal annual renewal line may be regarded as an asymptote.

Table 60. Values of Maximum and Minimum Annual Renewals for Seven Type Renewal Curves

Type Number	Magnitude of Maximum Annual Renewals	Magnitude of Minimum Annual Renewals
I	34.161	0.133
II	19.748	3.220
III	14.91	6.844
IV	12.74	8.715
V	11.575	9.302
VI	10.930	9.674
VII	10.522	9.723

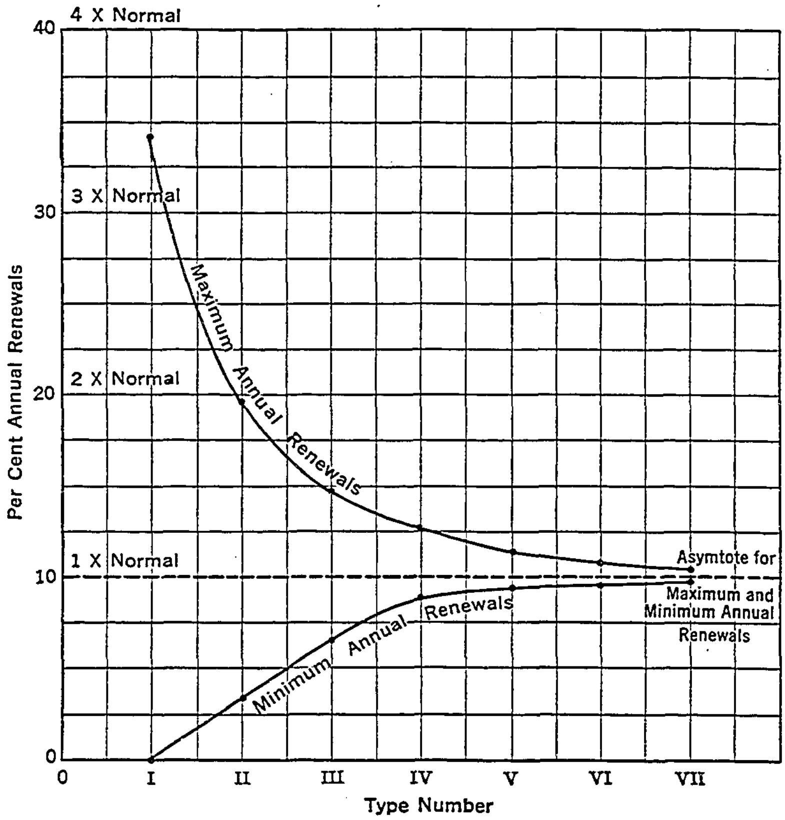


Figure 90. Variation in Maximum and Minimum Annual Renewals with Types

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